SOME IMBEDDING THEOREMS AND CHARACTERIZATION PROBLEMS OF DISTANCE GEOMETRY

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Introduction. The systematic development of abstract distance geometry was initiated by Menger's *Untersuchungen* [6][1] of 1928. The field opened up by these pioneering papers has been extensively cultivated during the past fourteen years with the result that not only have the boundaries of the subject been extended far beyond what was envisaged a decade and a half ago, but the territory gained has been regained and consolidated by new methods that attain their objectives more easily. It is the purpose of this paper to present some of these consolidations (Part I) and extensions (Part II)—at hand or in progress—and to point out a few promising regions for future exploration.

PART I

1. **Preliminary definitions.** A *distance space*, in its most general aspects, arises upon associating with each ordered pair p, q of elements of a "point" set an element pq of a "distance" set, the association being conditioned only by certain very simple rules. If, in particular, the distance set is the set of non-negative real numbers, $pq=qp$, while $pq=0$ if and only if $p=q$, one obtains the class $\{\Sigma\}$ of *semimetric spaces*. A semimetric space is *metric* provided that for each three of its points p, q, r the triangle inequality $pq+qr \geqq pr$ is satisfied.

We shall have frequent occasion to consider the determinant

$$D(p_1, p_2, \cdots, p_k) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_1p_2^2 & \cdots & p_1p_k^2 \\ 1 & p_2p_1^2 & 0 & \cdots & p_2p_k^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & p_kp_1^2 & p_kp_2^2 & \cdots & 0 \end{vmatrix},$$

formed for k points $p_1, p_2, \cdots, p_k$ of a semimetric space. With its use the triangle inequality takes the symmetric form $D(p, q, r) \leqq 0$.

Two distance spaces (or subsets of the same distance space) are

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[1] See references at end of paper.

called *congruent* or *isometric* provided there exists a one-to-one, distance-preserving correspondence between their points, and a *distance geometry* studies those properties of a distance space which are invariant under the group of congruences.

2. **Problems for euclidean subsets and spaces.** It is observed that a semimetric space of m elements $p_1, p_2, \cdots, p_m$ is given by a matrix (p_ip_j) of non-negative real numbers—the mutual distances of the m points—which, in general, is subject only to the requirements of being symmetric and having zeros along the principal diagonal. It is clear that if such a distance matrix be formed for m points of a particular space (for example, euclidean) there is a marked decrease in the freedom with which the elements of the matrix may be selected. The distinctive character of the space is reflected in the structure (the inner relations) of this matrix, and it is natural to seek to classify and study distance spaces in terms of this structure.[2] The first problems to which these considerations gave rise were the following ones.

THE EUCLIDEAN SUBSET PROBLEM. *What is the structure of the distance matrix of a finite semimetric space which is necessary and sufficient to insure the euclidean character of the metric? On what finite subsets (if any) of an arbitrary semimetric Σ is it necessary and sufficient to impose this matrix structure in order that Σ may be congruently contained in a euclidean space of a given dimension?*[3]

THE EUCLIDEAN SPACE PROBLEM. *Let Σ be any semimetric space. What are necessary and sufficient (metric) conditions for congruence of Σ with a euclidean space of given dimension?*

The first solutions of both of these problems were given by Menger [6].

It is important to observe that the subset problem is the more general one. Of the many solutions that have been given for the space problem few advance the solution of the problem of congruent imbedding. This seems due to the fact that in characterizing a space one

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[2] There are many ways in which this structure may be described. As seen later, one may base the description upon (1) the signs of principal minors of $D(p_1, \cdots, p_m)$ (Menger), (2) the character of the quadratic form with determinant D (Morse), (3) the character of the quadratic form $(1/2)\sum_2^m(p_1p_i^2+p_1p_j^2-p_ip_j^2)x_ix_j$ (Schoenberg), (4) the behavior of the family of functions $\exp(-\lambda t^2)$, λ positive, defined over the space of m elements (E. H. Moore, Schoenberg), and so on. It would be useful to add to these means of interpreting the structure.

[3] It should be mentioned that this formulation of the euclidean subset problem is quite different from the original one.

imposes from the outset certain obvious necessary conditions that profoundly affect the type of theorem obtained. So far as imbeddability is concerned, such theorems give only stringent sufficient conditions.

Thus Busemann has shown that a convex, externally convex, finitely compact metric space is congruent to a euclidean or hyperbolic space provided the locus of points with equal distances from two distinct points contains with each distinct pair of its points each straight line through them.[4] This interesting result is, however, of little use in determining if an arbitrary semimetric space is congruently contained in a euclidean or hyperbolic space.

On the other hand, a solution of the euclidean subset problem does advance the solution of the space problem, for if subsets of E_n are characterized metrically a characterization of E_n itself is obtained by adjoining properties that distinguish E_n among its subsets. This is a *euclidean* rather than an abstract space problem. It should be mentioned that a characterization of the E_n by way of the subset problem is quite likely to contain redundancies in the hypotheses, for it may happen that some of the requirements which are essential for the imbedding of arbitrary semimetric spaces in E_n might be dispensed with in the light of those adjoined conditions individualizing the E_n among its subsets. We shall see an illustration of this later.

Before turning to the solutions of the two problems posed above, we state here the general characterization problem.

GENERAL CHARACTERIZATION PROBLEM. *Let* $\{\Delta\}$ *be a given class of distance spaces and* $\{\Delta^*\}$ *a given subclass of* $\{\Delta\}$. *The subclass* $\{\Delta^*\}$ *is characterized metrically with respect to the class* $\{\Delta\}$ *when metric conditions are obtained which are necessary and sufficient to insure that any space of* $\{\Delta\}$ *satisfying them be congruent with a member of* $\{\Delta^*\}$.

The problem of *isometric or congruent imbedding* consists in determining whether a given space Δ be congruently contained in a given space Δ^* (that is, congruent with a subset of Δ^*). The congruence of the spaces $L^{(2)}$ and $l^{(2)}$ (1907), the isometric imbedding in Urysohn's space U (or the space C of continuous functions on (0, 1)) of each separable metric space are examples.

3. **Solution of euclidean subset and space problems.** The problem of congruent imbedding of Σ in E_n is reduced to a "finite" one by

[4] For definitions of convexity, external convexity, and so on, see the writer's *Distance geometries*, University of Missouri Studies, vol. 13 (1938), which presents a survey of the development of abstract metrics.

showing that *any semimetric space is congruently contained in E_n whenever each $n+3$ of its points are.* This is described by saying that the E_n has *congruence order* $n+3$ with respect to the class of semimetric spaces. In the following we give the imbedding theorems with weaker hypotheses than those originally imposed.[5]

The congruence order $n+3$ property of the E_n is a consequence of the following easily proved theorem:

THEOREM 3.1. *A semimetric Σ is congruently contained in E_n if and only if there exists an integer r, $r \leqq n$, such that* (1) *Σ contains $r+1$ points congruently imbeddable in E_r, not in E_{r-1}, and* (2) *each $(r+3)$-tuple of Σ containing this $(r+1)$-tuple is imbeddable in E_r. Then Σ is congruently contained in E_r, not in E_{r-1}.*

This theorem differs principally from Menger's congruence order theorem by not assuming that all $(r+3)$-tuples are imbeddable in E_r, but only those that contain a selected set of $r+1$ points. It thus permits the presence of *free* $(r+3)$-tuples; that is, sets of $r+3$ points which are not assumed imbeddable in E_r. We shall meet with other instances of this kind of freedom later. Further, the proof of the property as formulated above dispenses with a separation into cases and an induction which feature Menger's proof.

The euclidean subset problem is then solved when conditions for (1) the imbedding of $r+1$ points in E_r, not in E_{r-1}, and (2) the imbedding in E_r of $(r+3)$-tuples containing such a set of $r+1$ points are obtained. Concerning (1) we have this statement:

THEOREM 3.2. *A semimetric $(r+1)$-tuple $p_1, p_2, \cdots, p_{r+1}$ is imbeddable in E_r, not in E_{r-1}, if and only if*

$$\operatorname{sgn} D(p_1, p_2, \cdots, p_{k+1}) = (-1)^{k+1}, \qquad k = 1, 2, \cdots, r.$$

This theorem was originally proved by Menger under the heavier assumptions that for each integer k $(2 \leqq k \leqq r+1)$ and for each set of k of the $r+1$ points, $\operatorname{sgn} D(p_{i_1}, p_{i_2}, \cdots, p_{i_k}) = (-1)^k$. Menger later observed that as a consequence of a remark of M. Morse it suffices to require merely the non-vanishing of the determinants of every k-tuple $(2 \leqq k \leqq r+1)$ of the points and $\operatorname{sgn} D(p_1, p_2, \cdots, p_{k+1}) = (-1)^{k+1}$ $(k = 1, 2, \cdots, r)$ [5]. It turns out, however, that even the non-vanishing of these determinants need not be assumed.[6]

[5] In a recent seminar course, the writer has completely revised Menger's treatment of the euclidean problems and obtained simpler proofs and stronger theorems. This applies particularly to the difficult matter of quasi-congruence order.

[6] This is implied also by Schoenberg's quadratic form criterion, but we shall use these results to obtain Schoenberg's theorem.

COROLLARY. *If $p_1, p_2, \cdots, p_r$ are congruently contained in E_{r-1}, not in E_{r-2}, then the semimetric $(r+1)$-tuple $p_1, p_2, \cdots, p_r, p_{r+1}$ is imbeddable in E_r, not in E_{r-1}, if and only if* $\operatorname{sgn} D(p_1, p_2, \cdots, p_r, p_{r+1}) = (-1)^{r+1}$.

It is of interest that the inequality demanded of the determinant D of the $r+1$ points is strong enough to force the selection of the r distances of p_{r+1} from the remaining r points so that the r-tuples $p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_r, p_{r+1}$ $(i=1, 2, \cdots, r)$ are euclidean. Turning to the imbedding of $r+3$ points, we have this theorem.

THEOREM 3.3. *If $p_1, \cdots, p_{r+1}$ are imbeddable in E_r, not in E_{r-1}, then the semimetric $(r+3)$-tuple $p_1, \cdots, p_{r+1}, p_{r+2}, p_{r+3}$ is imbeddable in E_r if and only if*

$$D(p_1, \cdots, p_{r+1}, p_{r+2}) = D(p_1, \cdots, p_{r+1}, p_{r+3}) = D(p_1, \cdots, p_{r+1}, p_{r+2}, p_{r+3}) = 0.$$

The proof of this theorem follows familiar lines when it is established that $p_1, p_2, \cdots, p_{r+1}, p_{r+2}$ and $p_1, p_2, \cdots, p_{r+1}, p_{r+3}$ are imbeddable in E_r, not in E_{r-1}. This is done in a lemma, and the euclidean subset problem is completely solved.[7]

By despoiling $D(p_0, p_1, \cdots, p_k)$ of its bordering, we have

$$D(p_0, p_1, \cdots, p_k) = (-1)^{k+1} \cdot 2^k \left| p_{ij} \right|, \qquad i, j = 1, 2, \cdots, k,$$

where $p_{ij} = (1/2)(p_0 p_i^2 + p_0 p_j^2 - p_i p_j^2)$. It follows at once from the preceding theorems that a semimetric $(n+1)$-tuple $p_0, p_1, \cdots, p_n$ is imbeddable in E_n if and only if there exists an integer r, $r \leq n$, such that for $r+1$ of the $n+1$ points, say $p_0, p_1, \cdots, p_r$, $\left| p_{ij} \right| > 0$ $(i, j=1, 2, \cdots, k; k \leq r)$ while in case $r<n$, all $(r+1)$st and $(r+2)$d order determinants $\left| p_{ij} \right|$ containing $\left| p_{ij} \right|$ $(i, j=1, 2, \cdots, r)$ vanish. *These are precisely the conditions that the quadratic form $\sum_{i,j=1}^{n} p_{ij} x_i x_j$ be positive definite of rank r, and we have the theorem of Schoenberg* [10]. It is easy to see that this condition is equivalent to the form $\sum_{i,j=0}^{n} (p_i p_j)^2 x_i x_j$ being negatively definite of rank r on the hyperplane $x_0 + x_1 + \cdots + x_n = 0$, and this form of the condition has proved to be very useful.

Turning again to the congruence order $n+3$ property of the E_n, it

[7] We see that if $p_1, \cdots, p_r$ are imbeddable in E_{r-1}, not in E_{r-2}, then the semimetric $(r+1)$-tuple $p_1, \cdots, p_r, p_{r+1}$ is imbeddable in E_r if and only if the sign of their determinant D is *not* $(-1)^r$. An example shows this is no longer valid if the restriction that the r points be not imbeddable in E_{r-2} is dropped; for if $pq = qr = ps = 1$, $pr = 2$, $qs = 3$, $rs = (19)^{1/2}$, the triple p, q, r is imbeddable in E_2 and $D(p, q, r, s) = 0$, but the four points are not congruently contained in a euclidean space.

is natural to inquire into the conditions that permit a reduction in the number $n+3$. That this number is in general the smallest integer k for which the imbedding of each k-tuple of a semimetric space implies the imbedding of the whole space in E_n follows, as Menger has shown, from the existence for each integer n of semimetric spaces of $n+3$ points which are not imbeddable in E_n though each $(n+2)$-tuple is.

The construction of these so-called pseudo-E_n $(n+3)$-tuples is readily described, but only in the case of pseudo-linear quadruples $(n=1)$ have distance relations characterizing such sets been obtained. These quadruples have the form $pq=rs=a>0$, $qr=ps=b>0$, $pr=qs=a+b$, and $D(p, q, r, s)=-32$ (product of the six distances). It would be of interest to find the value of D for a pseudo-E_n $(n+3)$-tuple. So far it has been shown only that its sign is $(-1)^n$.

It follows from the preceding remark that pseudo-E_n sets of $n+3$ points are not imbeddable in any euclidean (or Hilbert) space. Pseudo-linear quadruples are clearly imbeddable in a (convex) circle, but pseudo-planar quintuples are not imbeddable, so far as is known, in any singularity-free surface.[8] Recently spaces were constructed by Pepper which contain congruently all pseudo-E_n $(n+3)$-tuples [7]. They are obtained by appropriately metrizing the union of two n-spaces "joined" along a simplex. For $n=2$ a simpler kind of space suffices. This is obtained by metrizing convexly the set consisting of the union of three half-planes with a common axis.

What of pseudo-E_n sets of more than $n+3$ points? The most complicated part of the *Zweite Untersuchung* is devoted to showing that no such sets exist; that is, *if a semimetric space has more than $n+3$ points it is imbeddable in E_n whenever each $n+2$ of its points has this property.* This is described by saying that the E_n has *quasi-congruence order* $n+2$ with respect to the class of semimetric spaces. We shall return to this matter later.

Having solved the euclidean subset problem, the characterization of the whole E_n is obtained by adjoining properties distinguishing it from its subsets. This yielded the following theorem.

THEOREM 3.4. *A semimetric space Σ is congruent to E_n if and only if Σ is complete, convex, externally convex, has each $n+2$ of its points imbeddable in E_n and at least one $(n+1)$-tuple not imbeddable in E_{n-1}.*

With this the principal objectives of the *Zweite Untersuchung* are attained. The reader is referred to *Distance geometries* for a discussion

[8] The convex circle is the circle (circumference) with shorter arc metric.

of the results of W. A. Wilson who, attacking the space problem directly, obtained a characterization theorem in terms of the imbedding of *quadruples* instead of $(n+2)$-tuples, as well as for the work of Aronszajn bearing upon the space problem.[9] By weakening Wilson's four-point property to require merely that each quadruple containing a linear triple be imbeddable (*weak four-point property*) the writer allowed the existence of *free* quadruples. Very recently B. E. Gillam characterized the E_3 by showing that a complete, convex, externally convex metric space with the weak four-point property, which contains at least one quadruple with non-vanishing determinant D, while D vanishes for each five points of the space, *is logically equivalent to* E_3 [4]. This was accomplished by proving all of Hilbert's postulates for E_3 on the basis of the above assumptions, with lines, planes, betweenness and congruence appropriately defined.

PART II

Foreword. As indicated in the introduction, this part of the paper is primarily concerned with extensions rather than consolidations. I propose to take up some recently established imbedding and characterization theorems, along with pertinent notions and concepts, and to raise certain questions suggested by them. No attempt is made to catalogue all of the new results; on the contrary, I limit myself to the discussion of the very few with which I have been connected.

1. **Concerning quasi-congruence order and free m-tuples.** We have seen that one of the important metric properties of the E_n is its possession of quasi-congruence order $n+2$ with respect to semimetric spaces; that is, any semimetric space of more than $n+3$ distinct points is congruently contained in the E_n whenever *each* $n+2$ of its points are. Several kinds of inquiries are suggested by this interesting concept.

(I). It has been observed in Part I that the requirement featuring the notion of congruence order $n+3$ (namely, that *each* $(n+3)$-tuple be imbeddable in E_n) has been weakened by allowing the presence of free $(n+3)$-tuples (that is, $(n+3)$-tuples which are not assumed congruently imbeddable in E_n) and the question arises whether a similar kind of weakening might be allowable here. Is the imbedding still valid if a certain number of the $(n+2)$-tuples are free? If so, how many of the $(n+2)$-tuples might be taken as free?

[9] It should be noted, however, that Wilson's work is not entirely independent of the subset problem for it utilizes Menger's theorem on the imbedding of $n+1$ points in E_n.

In answering these questions it is first of all clear that it does not suffice to require merely that each $(n+2)$-tuple containing a selected set of $n+1$ independent points be imbeddable in E_n—a type of weakening immediately suggested by what proved to be effective in the case of congruence order. If $n=1$, for example, this demands only that every triple containing a given pair of distinct points be linear (that is, imbeddable in E_1). That the linearity of a space does not follow from this demand is seen by considering the convex circle, which is not imbeddable in the E_1 though each of its triples containing a given pair of diametral points is linear. There is, however, an analogue of the previous type of weakening. It is given in the following theorem.

THEOREM 1.1. *Let Σ be a semimetric space of at least $n+4$ distinct points. Σ is congruently contained in the E_n, not the E_{n-1}, if and only if* (a) *Σ contains a set S of $n+2$ distinct points of which at least one $(n+1)$-tuple is not imbeddable in E_{n-1}, and* (b) *each $n+2$ points of Σ with at least n points in common with S is imbeddable in E_n.*

PROOF. Let $p_1, p_2, \cdots, p_{n+1}, p_{n+2}$ denote the points of S, with the first $n+1$ not imbeddable in E_{n-1}. Using (b) it follows that these $n+1$ points are imbeddable in E_n. If, now, $p_1, p_2, \cdots, p_{n+1}, r, s$ is any $(n+3)$-tuple of distinct points containing the independent $n+1$ points, consider the $n+4$ points formed by annexing the point p_{n+2}. (The case where either r or s is p_{n+2} causes no difficulty.) By (b) each $n+2$ of these $n+4$ points is congruently contained in E_n, and hence, by the quasi-congruence order $n+2$ property of E_n, the $n+4$ points are imbeddable in E_n. Thus each $(n+3)$-tuple containing the independent $(n+1)$-tuple $p_1, p_2, \cdots, p_{n+1}$ is imbeddable in E_n and it follows that the whole space is congruently contained in E_n.

An example shows that this theorem is no longer valid if the assumption that at least one $(n+1)$-tuple of S be not imbeddable in E_{n-1} is not made, suppressing at the same time, of course, the requirement that Σ be not imbeddable in E_{n-1}. Let p_1, p_2, p_3, form a plane equilateral triple and label the in-center p_4 and p_5. The semimetric space Σ whose points are p_3, p_4, p_5 together with the points of the straight line joining p_1 and p_2, with all distances euclidean *except the distance p_4p_5*, which is defined to be the radius of the circumcircle of p_1, p_2, p_3, contains a subset S of four distinct points $(p_1, p_2, x, y,$ where x, y are any two points of the line other than $p_1, p_2)$ such that every four points of Σ with at least two points in common with S are imbeddable in the plane E_2. But the space Σ is obviously not imbeddable in E_2.

A more striking way of lifting the demand that every $(n+2)$-tuple be imbeddable in E_n is given by the following theorems.

THEOREM 1.2. *Let Σ be semimetric with at least $n+k+4$ distinct points. Then Σ is imbeddable in E_n if and only if Σ contains at most k free $(n+2)$-tuples.*

PROOF. We shall show that each $(n+2)$-tuple of Σ is congruently contained in E_n.

Let, then, $p_1, p_2, \cdots, p_{n+2}$ be a free $(n+2)$-tuple. Since there are at most k free $(n+2)$-tuples and at least $n+k+4$ points in Σ it is readily seen that this $(n+2)$-tuple is contained in an $(n+5)$-tuple which has this $(n+2)$-tuple as its only free set of $n+2$ points. If $p_1, \cdots, p_{n+5}$ is such an $(n+5)$-tuple, each of the three sets of $n+4$ of these points obtained by omitting p_n, p_{n+1}, p_{n+2} in turn have all $(n+2)$-tuples imbeddable in E_n, and hence the three $(n+4)$-tuples are themselves imbeddable in E_n. It follows that the rank of the determinant D of the $n+5$ points is at most $n+2$, and so $D(p_1, p_2, \cdots, p_n, p_{n+1}, p_{n+2})=0$.

Now each $n+1$ points of Σ are imbeddable in E_n, for each $(n+1)$-tuple is contained in at least $k+3$ $(n+2)$-tuples and not all of these are free since, by hypothesis, Σ contains at most k free $(n+2)$-tuples. Thus, each $n+1$ of the points $p_1, p_2, \cdots, p_{n+2}$ is imbeddable in E_n. Since the determinant D of these $n+2$ points is zero, it follows that the points are imbeddable in E_n, and the theorem is proved.

The same kind of argument yields the next theorem.

THEOREM 1.3. *Let Σ be semimetric with power exceeding $\aleph_0$. Then Σ is congruently contained in E_n if and only if Σ has at most $\aleph_0$ free $(n+2)$-tuples.*

(II). The literature contains only two proofs of the quasi-congruence order $n+2$ property of the E_n, both of which are quite lengthy. A direct elementary proof is desired. Since the property is easily formulated in terms of determinant or quadratic form theory, attempts have been made to establish the result purely algebraically. For $n=1$, not a trivial case, a very simple proof has been found by the writer, but in general such efforts have so far not been effective. Since the complex space K_2 does not have quasi-congruence order 4 with respect to the class of complex distance spaces, a proof of the property for the E_n must depend upon the reality of the field from which the distance elements are selected and hence is likely to be difficult. On the other hand, the writer has used algebraic methods to

materially shorten the two earlier proofs. The question is, it seems, still open.

(III). It would be highly desirable to axiomatize the notion of quasi-congruence order or at least to express it in terms of more elementary properties of the space. A start in this direction has been made by C. V. Robinson and the writer by showing that for *complete, convex, externally convex metric spaces quasi-congruence order three (with respect to the class of semimetric spaces) is equivalent to the absence of equilateral triples of distinct points.* This was done by establishing the following very simple metric characterization of the line [3].

THEOREM 1.4. *A complete, convex, externally convex metric space, with at least two points, is a straight line if and only if it does not contain an equilateral triple of distinct points.*

This theorem has as immediate corollaries characterizations of the line due to Lindenbaum and to Menger.

More recently the writer showed that *a line segment is characterized among all compact and convex metric spaces by the absence of equilateral triples.*

Quasi-congruence order $n+2$ readily implies the absence of equilateral $(n+2)$-tuples, but not conversely. The half-line, for example, has neither equilateral triples nor quasi-congruence order three, and the K_2 has neither equilateral quadruples (with nonzero side) nor quasi-congruence order 4. *What is a large class of spaces for which the two notions are logically equivalent?*

In this connection, I would like to raise the question of interpreting *topologically* the absence of equilateral k-tuples. Biedermann has shown that a connected metric space in which each triple is linear (that is, a space *strongly* without equilateral triples) is homeomorphic with a line, a ray, or a segment. It was in fact this result that motivated Menger's characterization of euclidean subsets. (Menger showed that the homeomorphism of Biedermann is actually a congruence.)

If, now, one supposes merely that a connected metric space is without equilateral triples, what topological properties of the space ensue? *Is a metric Peano continuum without equilateral triples an arc?* If this is indeed the case (and I conjecture that it is) a topological property is given an elegant metric characterization.[10] Such problems are part of a general *program* which seeks to determine *what topologi-*

[10] The writer has now established this conjecture. See *New characterizations of segments and arcs*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1943) pp. 107–109.

cal properties a space must possess in order that it may be homeomorphic to a space with given metric properties.

2. **Congruence indices.** Increasing study of characterization problems has yielded results that the concepts of congruence and quasi-congruence orders are inadequate to describe. To meet this situation the notion of relative congruence indices was formulated [1]:

Let Γ *and* Γ^* *be two given spaces and* $\{\Delta\}$ *a given class of spaces. The space* Γ *has* Γ^**-relative congruence indices* (n, k) *with respect to the class* $\{\Delta\}$ *provided any space* Δ *of the class with more than* $n+k$ *distinct points is imbeddable in* Γ *whenever each* n *of its points (not necessarily distinct) is imbeddable in* Γ^*.

The space Γ^* is called Γ-*catalytic* of indices (n, k) with respect to $\{\Delta\}$ since it (in general) facilitates the desired reaction of imbedding Δ in Γ without participating in the result, for Δ is not necessarily imbeddable in Γ^*. If $\Gamma \equiv \Gamma^*$ the indices (n, k) are called *congruence indices* of Γ with respect to the class $\{\Delta\}$.

It is easy to see that congruence and quasi-congruence orders correspond to congruence indices $(k, 0)$ and $(k, 1)$, respectively, and that if Γ has Γ^*-relative congruence indices (n, k), then Γ has Γ^*-relative indices (n', k') if $n \leqq n'$ and $n+k \leqq n'+k'$, the class of comparison spaces being fixed. We order these indices lexicographically, and call those indices (n, k) "best" which are not preceded by any indices (n', k'). It is important to observe that this ordering is *not* that of logical implication, for with respect to the latter criterion the indices do not form an ordered set. Thus a two-dimensional spherical cap of radius less than $\pi r/2$ has, with respect to the class of all semimetric spaces, congruence indices (4, 2) and also indices (5, 0), but neither set logically implies the other.

Keeping the comparison class of spaces fixed, let the best congruence indices of a space Γ be (n, k), with (n', k') the best Γ^*-relative congruence indices. The space Γ^* is *strongly* or *weakly* catalytic according as $n' < n$ or $n' = n$, $k' < k$, respectively. In case the indices (n', k') are no better than (n, k), Γ^* is *formally* catalytic, while if $n' < n$ and $k' < k$, Γ^* is *perfectly* catalytic. An open hemisphere of $S_{2,r}$, for example, is a strongly catalytic space for $S_{2,r}$, since by its use the best congruence indices (5, 0) of $S_{2,r}$ with respect to semimetric spaces are bettered by the hemisphere-relative congruence indices (4, 2). In fact, it has been shown that the $S_{n,r}$[11] has Γ^*-relative congruence indices $(n+2, n)$ with respect to all semimetric spaces,

[11] $S_{n,r}$ denotes the n-dimensional surface of a sphere of radius r in E_{n+1}, with geodesic (shorter great circle arc) metric.

where Γ^* denotes an open (n-dimensional) hemisphere of the $S_{n,r}$.

It is worth noting that pseudo-euclidean spaces fall under the notion of catalytic spaces. Clearly every subset of a catalytic space is catalytic (with the same indices) but the interest lies in enlarging rather than depleting these spaces, and the problem of finding saturated catalytic spaces of given indices arises.

The writer recently proposed the problems of finding the best congruence indices of the (closed) n-dimensional hemisphere and of small n-dimensional spherical caps with respect to (1) the class of semimetric spaces and (2) the set of all subsets of the containing $S_{n,r}$. The second inquiry led to posing the following question: *for what integer k does the intersection of each k members of a family of convex subsets of the n-dimensional spherical surface $S_{n,r}$ imply the existence of a common point for the family*?

In his Missouri dissertation, C. V. Robinson investigated these problems [9]. He showed that the best congruence indices of the n-dimensional hemisphere with respect to semimetric spaces are $(2n+1, 1)$. Even with respect to subsets of the containing $S_{n,r}$ these indices cannot be bettered. It follows that any subset S of $S_{n,r}$ is coverable by a hemi-$S_{n,r}$ provided S contains more than $2n+2$ distinct points and each $2n+1$ of its points is coverable by the hemi-$S_{n,r}$.

Of particular interest is the plane analogue of a cap theorem to the effect that a circular disc will cover a plane set P if and only if each three points of P are coverable by the disc. This behavior of the circular disc is all the more striking when contrasted with that of other convex subsets of the plane such as the square or elliptical disc. *For neither of these figures does any such integer k exist*; for example, for any integer k however large there are plane sets not coverable by a square (elliptical) disc even though each k points of the set are so coverable. The same is true for any portion of the plane bounded by a broken-line curve. The possession of indices (3, 0) with respect to subsets of the plane has been made the basis of a characterization of the circular disc [8].

THEOREM 2.1. *The circular disc is the only connected, simply connected domain (closure of a bounded open set) of the plane with congruence indices* (3, 0) *with respect to plane sets.*

It would be highly desirable to make an exhaustive classification of plane sets according to their congruence indices with respect to subsets of the plane. The problems involved seem quite difficult and only the beginnings of a systematic investigation have been made. With

respect to the class of linear subsets we have a partial enumeration of linear sets with fixed congruence orders (finite, hyperfinite, and transfinite).[12] Investigations aimed at completing these results and developing a similar theory for plane subsets are now in progress.

In this connection interest is attached to a new type of problem recently considered which I illustrate here for the simplest case. It is easily seen that corresponding to each integer k there exists a bounded and closed subset of E_1 with best (finite) congruence order exceeding k. Thus if P is an arbitrary bounded and closed linear set, one may not assert that any linear set Q is imbeddable in P whenever each k of its points are, even for k arbitrarily large. Note that, as is usual in imbedding theorems, the condition "every k points of Q are imbeddable in P" is a *unilateral* one. Suppose, on the other hand, it is also assumed that *every k points of P are imbeddable in Q*. Then one easily proves that P and Q are congruent when $k=4$; that is, *if P and Q are linear sets with P or Q bounded and closed, then $P \approx Q$ if and only if each four points of P are imbeddable in Q and each four points of Q are imbeddable in P*. This bilateral type of condition gives rise to several new problems and causes the notion of congruence order (based upon unilateral conditions) to be viewed in a somewhat different light.

In answering the question raised concerning intersections of convex subsets of $S_{n,r}$, the following theorems were obtained.[13]

THEOREM 2.2. *If each $n+k+2$ members of a family of convex subsets of the sphere $S_{n,r}$ intersect, and if one member contains no $S_{k,r}$ $(0 \leqq k \leqq n)$, then there is a point common to all.*

THEOREM 2.3. *A family of more than $2n+2$ convex subsets of $S_{n,r}$ has a common point if each $2n+1$ of them intersect.*

Additional theorems are obtained when the diameters of the convex subsets are subject to certain restrictions.

[12] A space Σ has hyperfinite congruence order with respect to a class of spaces $\{\Delta\}$ if and only if each member of the class is imbeddable in Σ whenever each of its finite subsets are. Thus hyperfinite congruence order is intermediate between finite and transfinite congruence orders. Each self-compact linear set has hyperfinite congruence order with respect to semimetric spaces; examples have been given of linear sets with hyperfinite and no finite congruence order, and transfinite but not hyperfinite congruence order. The complement of a finite or denumerable linear set has best congruence order c (the power of the continuum). These notions were introduced in the Missouri dissertation of C. V. Robinson.

[13] A subset of $S_{n,r}$ is convex provided it is a product of hemispheres. This definition rules out the whole $S_{n,r}$ and includes a pair of diametral points. Thus from Theorem 2.2 it follows that a family of convex subsets of $S_{n,r}$ has a common point whenever each $2n+2$ of them intersect.

Intersection theorems for convex subsets of E_n were obtained by Helly. Due to the variety of results such problems on the sphere turn out to be more interesting than their analogues in euclidean spaces. In contrast, for example, to the Helly theorem in E_2 which states that any family of convex subsets has a non-empty product whenever each three members of the family intersect, it is found that a family of convex subsets of $S_{2,r}$ have a common point if each *six* have or (if the family has more than six members) if each *five* have, or (if one member of the family contains no diametral point-pairs) if each *four* have. If each of the convex subsets has spherical diameter less than $2\pi r/3$ it suffices for each three of the sets to have a common point. It is clear how these results may be translated into "covering" theorems for spherical caps.

3. **δ-supplementation and elliptic space.** If Σ is semimetric with finite diameter d, denote the set of *unordered* pairs of its elements by S^2, and let $S^2 = S_1 + S_2$, $S_1 \cdot S_2 = 0$, be any decomposition of S^2 into two mutually exclusive subsets. Corresponding to such a decomposition, a real, non-negative function $F(x)$ is defined:

$$F(x) = x, \qquad x = pq;\ (p, q) \in S_1,$$

$$F(x) = \delta - x, \qquad x = pq;\ (p, q) \in S_2,$$

where δ is a fixed real number not less than d. Transforming metrically the space Σ by this function F yields a space which is in general not semimetric.[14] Nor does it necessarily become semimetric upon identification of those points p, q (and only those) with $pq^* = F(pq) = 0$, for such an identification is justified (keeping to a single-valued metric) if and only if for each element r of the space $pr^* = qr^*$.

If, however, $\delta > d$, such an identification does convert the transformed space into a semimetric one whose point set may, on account of the identification, differ from that of Σ. We denote such a space by Σ^* and refer to it as a δ-supplement of Σ (more precisely, *the* δ_F-supplement of Σ). We write $\Sigma^* = \sup_\delta \Sigma$ when there is no need to specify the function F.

An important example of a δ-supplement is obtained by taking the

[14] A space $F(\Sigma)$ which arises from Σ by replacing the metric pq of Σ by $F(pq)$, where F is a real, single-valued, non-negative function defined for every $x = pq$, $p, q \in \Sigma$, is called the metric transform of Σ by F. A study of metric transforms was begun by the writer in 1934 (Ergebnisse eines der Mathematischen Kolloquiums, Wien, no. 7, pp. 8–10). Metric transforms of euclidean spaces into subsets of Hilbert space have been extensively investigated by Schoenberg and von Neumann (1938, 1942).

n-dimensional convex spherical surface $S_{n,r}$ as Σ, and transforming it metrically by the function F.

$$(\dagger) \qquad \begin{aligned} F(x) &= x, & x &= pq;\ pq \leqq \pi r/2, \\ F(x) &= \pi r - x, & x &= pq;\ pq > \pi r/2. \end{aligned}$$

(Here $\delta = d$.) The resulting semimetric space is the *n-dimensional elliptic space $\mathcal{E}_{n,r}$ with space constant r.*

In work still in progress, the notion of δ-supplementation is made the basis for characterizing subsets of $\mathcal{E}_{n,r}$ among all semimetric spaces [2]. Elliptic space differs profoundly (both metrically and topologically) from all spaces heretofore treated and these differences make the older methods either inapplicable or extremely tedious. Fundamental in our work is the following theorem.

THEOREM 3.1. *A semimetric Σ is congruently contained in $\mathcal{E}_{n,r}$ if and only if* (i) *no distance in Σ exceeds $\pi r/2$ and* (ii) *there exists at least one function F such that the δ_F-supplement of Σ is imbeddable congruently in $S_{n,r}$, where $\delta = \pi r$.*

PROOF. Let Σ be congruently contained in $\mathcal{E}_{n,r}$. Then evidently $pq \leqq \pi r/2$ for each pair p, q of elements of Σ. Denoting the metric transform of $S_{n,r}$ by the function F defined in (†) by $\sup_{\pi r} S_{n,r}$, we have $\mathcal{E}_{n,r} = \sup_{\pi r} S_{n,r}$. Evidently

$$\text{subset } \mathcal{E}_{n,r} = \text{subset}(\sup_{\pi r} S_{n,r}) = \sup_{\pi r}(\text{subset } S_{n,r}),$$

and hence there exists a sup(subset $\mathcal{E}_{n,r}$) which is a subset of $S_{n,r}$. Since, now, Σ is congruently contained in $\mathcal{E}_{n,r}$, there is a subset of $\mathcal{E}_{n,r}$ congruent with Σ, and, by the above, a supplement of this subset is a subset of $S_{n,r}$. Thus, a supplement of Σ is imbeddable in $S_{n,r}$ and the necessity is proved.

On the other hand suppose that Σ is semimetric with distances not exceeding $\pi r/2$ and that some πr-supplement of Σ is congruently contained in $S_{n,r}$. Denote any such supplement by sup Σ. Then there exists a subset $S'_{n,r}$ of $S_{n,r}$ such that sup Σ is congruent with $S'_{n,r}$. Now $\sup_{\pi r} S'_{n,r} = \text{subset } \mathcal{E}_{n,r}$ and hence $\sup_{\pi r}(\sup \Sigma)$ is congruently contained in $\mathcal{E}_{n,r}$.

The proof is complete when it is shown that for every sup Σ, $\sup_{\pi r}(\sup \Sigma)$ is identical with Σ. To this end, denote distances in sup Σ by $d(p, q)$ and in $\sup_{\pi r}(\sup \Sigma)$ by $d_{\pi r}(p, q)$. The same argument shows that the two pointsets and the two distance-sets are identical, for let $p = q$ in Σ. Then $pq = 0$. If $d(p, q) = 0$, then $d_{\pi r}(p, q) = 0$, while if $d(p, q) = \pi r$, then $d_{\pi r}(p, q) = 0$. Hence $p = q$ in Σ implies $p = q$ in

$\sup_{\pi r}(\sup \Sigma)$. Finally, if p and q are distinct points of Σ, then $0<pq \leqq \pi r/2$. If $d(p, q)=pq$ then, since $pq \leqq \pi r/2$, $d_{\pi r}(p, q)=pq>0$; while if $d(p, q)=\pi r-pq \geqq \pi r/2$, then $d_{\pi r}(p, q)=\pi r-(\pi r-pq)=pq>0$, and the theorem is proved.

By virtue of this theorem, imbedding in $\mathcal{E}_{n,r}$ is referred to imbedding πr-supplements of semimetric spaces in $S_{n,r}$. In any such supplement the distance pq of two points of Σ is either unchanged or replaced by $\pi r-pq$, and *hence* $\cos(d(p, q)/r)$ *differs from* $\cos(pq/r)$ *at most in sign.* Applying known imbedding theorems for the $S_{n,r}$ we have the following theorem.

THEOREM 3.2. *A semimetric m-tuple $p_1, p_2, \cdots, p_m$ is imbeddable in $\mathcal{E}_{n,r}$ if and only if* (i) $0<p_ip_j \leqq \pi r/2$ $(i, j=1, 2, \cdots, m; i \neq j)$ *and* (ii) *there exists an m-rowed symmetric square matrix $\epsilon=(\epsilon_{ij})$, $\epsilon_{ij}=\pm 1$, $\epsilon_{ii}=1$ $(i, j=1, 2, \cdots, m)$ such that the determinant $|\epsilon_{ij}\cos(p_ip_j/r)|$ $(i, j=1, 2, \cdots, m)$ has rank not exceeding $n+1$, with all nonvanishing principal minors positive.*

A determinant (symmetric) satisfying the condition (ii) of the theorem is said to have *positive ϵ-rank not exceeding $n+1$.*

Now it is easy to show that each separable semimetric space is congruently contained in $\mathcal{E}_{n,r}$ whenever each of its finite subsets has this property, and hence we have obtained necessary and sufficient conditions for the imbedding of any separable semimetric space in a given n-dimensional elliptic space. It is desirable, however, to establish reduction theorems which express the imbedding in terms of the congruent mapping of subsets of a fixed number of points. Such theorems hinge upon the following query.

QUERY. *What is the smallest integer k such that the determinant $|\cos(p_ip_j/r)|$ $(i, j=1, 2, \cdots, m)$ has positive ϵ-rank not exceeding $n+1$ whenever each of its principal minors of order k has positive ϵ-rank not exceeding $n+1$?*

It is conjectured that $k=n+3$, but so far *this has been proved only for* $n=1$.[15] Thus any separable semimetric space is imbeddable in $\mathcal{E}_{1,r}$ if and only if each four of its points have this property. This can be established in other ways, and without the hypothesis of separability. But for $n>1$, the algebraic method discussed here appears to be the only feasible approach. The above conjecture means (if verified)

[15] Examples show that k is not less than $n+3$.

that $\mathcal{E}_{n,r}$ has (like euclidean, hyperbolic and spherical spaces of n-dimensions) congruence order $n+3$. It would be interesting if the different topological properties of $\mathcal{E}_{n,r}$ would be made manifest by a difference in the congruence order.

4. **Concluding remarks.** In conclusion it should be observed that this paper has been concerned with completely *global* imbeddings and characterizations though interesting results in which local properties are assumed have also been obtained. A completely local type of theorem is one characterizing the line segment among all arcs of a metric ptolemaic space by the vanishing of the metrically defined Menger curvature at each point, while Gauss surfaces have been characterized among all compact and convex metric spaces by the existence at each point of a metrically defined surface curvature. A different ("mixed") type of imbedding theorem is expressed in terms of the local property of imbedding neighborhoods in E_n together with the global condition that each $n+3$ points be congruently contained in some euclidean space.

It has not been possible to present here any of the applications of metric imbedding theorems to other domains. Metric methods have already been successfully applied to many parts of analysis and algebra as well as to geometry, and there is reason to believe that continued cultivation of this field will abundantly increase both the number and the importance of these applications.

References

1. L. M. Blumenthal, *A new concept in distance geometry with applications to spherical subsets*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 435–443.

2. ———, *Metric characterization of n-dimensional elliptic space* $\mathcal{E}_{n,r}$, Bull. Amer. Math. Soc. abstract 48-1-75.

3. L. M. Blumenthal and C. V. Robinson, *A new characterization of the straight line*, Reports of a Mathematical Colloquium, issue 2 (1940) pp. 25–27.

4. B. E. Gillam, *A new set of postulates for euclidean geometry*, Revista de Ciencias vol. 42 (1940) pp. 869–899.

5. K. Menger, *Some applications of point-set methods*, Ann. of Math. (2) vol. 32 (1931) pp. 739–760. See in particular pp. 751, 752.

6. ———, *Untersuchungen über allgemeine Metrik*, I, II, III, Math. Ann. vol. 100 (1928) pp. 75–163. The *Zweite Untersuchung* is devoted to the characterization of euclidean subsets and spaces.

7. P. M. Pepper, *Nearly euclidean imbedding spaces for pseudo n-sets*, Reports of a Mathematical Colloquium, issue 3 (1941) pp. 34–46. See also issue 2 (1940), pp. 28–32.

8. C. V. Robinson, *A characterization of the disc*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 818, 819.

9. ——, *Spherical theorems of Helly type and congruence indices of spherical caps*, Amer. J. Math. vol. 64 (1942) pp. 260–272.

10. I. J. Schoenberg, *Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert,"* Ann. of Math. (2) vol. 36 (1935) pp. 724–732.

THE UNIVERSITY OF MISSOURI

THE FEBRUARY MEETING IN NEW YORK

The three hundred ninety-fourth meeting of the American Mathematical Society was held at Columbia University on Saturday, February 27, 1943. The attendance was approximately two hundred including the following one hundred sixty-seven members of the Society:

C. F. Adler, R. L. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, F. E. Baker, S. F. Barber, M. F. Becker, E. G. Begle, Stefan Bergman, Felix Bernstein, G. D. Birkhoff, Gertrude Blanch, W. M. Bond, C. B. Boyer, H. W. Brinkmann, A. B. Brown, Hobart Bushey, J. H. Bushey, S. S. Cairns, R. H. Cameron, M. D. Clement, L. M. Comer, T. F. Cope, Richard Courant, J. E. Crawford, H. B. Curry, M. D. Darkow, J. A. Daum, Norman Davids, A. S. Day, Jesse Douglas, Arnold Dresden, Nelson Dunford, J. E. Eaton, M. L. Elveback, Paul Erdös, J. M. Feld, W. K. Feller, A. D. Fialkow, Edward Fleisher, W. B. Fite, Tomlinson Fort, R. M. Foster, A. H. Fox, G. A. Foyle, Hans Fried, Orrin Frink, D. L. Fuller, R. E. Fullerton, H. P. Geiringer, B. H. Gere, David Gilbarg, B. P. Gill, A. M. Gleason, M. C. Gray, H. J. Greenberg, C. C. Grove, Laura Guggenbühl, Margaret Gurney, F. C. Hall, N. A. Hall, H. K. Hammer, Gerald Harrison, K. E. Hazard, Olaf Helmer, Einar Hille, T. R. Hollcroft, A. P. Holló, M. W. Hopkins, Harold Hotelling, S. E. Hotelling, E. M. Hull, Witold Hurewicz, W. H. Ingram, Nathan Jacobson, R. A. Johnson, H. A. Jordan, William Karush, Edward Kasner, S. C. Kleene, J. R. Kline, E. A. Knobelauch, B. O. Koopman, Mark Kormes, Arthur Korn, M. E. Ladue, Solomon Lefschetz, Marguerite Lehr, B. A. Lengyel, D. C. Lewis, Marie Litzinger, E. R. Lorch, Lee Lorch, A. N. Lowan, N. H. McCoy, R. S. McKee, C. C. MacDuffee, H. B. Mann, C. W. Mathews, A. E. Meder, Karl Menger, E. C. Molina, L. W. Miller, H. L. Mintzer, R. E. von Mises, A. K. Mitchell, Deane Montgomery, Parry Moon, Richard Morris, E. J. Moulton, F. J. Murray, C. A. Nelson, Otto Neugebauer, E. N. Nilson, C. O. Oakley, A. F. O'Neill, Oystein Ore, J. C. Oxtoby, E. L. Post, A. L. Putnam, L. L. Rauch, Moses Richardson, R. G. D. Richardson, C. E. Rickart, J. F. Ritt, E. K. Ritter, P. C. Rosenbloom, Raphael Salem, Arthur Sard, Henry Scheffé, I. J. Schoenberg, Alberta Schuettler, I. E. Segal, Seymour Sherman, Max Shiffman, James Singer, P. A. Smith, D. E. Spencer, V. E. Spencer, E. R. Stabler, W. J. Strange, W. C. Strodt, M. M. Sullivan, J. D. Tamarkin, Peter Treuenfels, A. W. Tucker, J. W. Tukey, D. F. Votaw, L. I. Wade, Abraham Wald, G. L. Walker, Wolfgang Wasow, Warren Weaver, André Weil, Louis Weisner, Hermann Weyl, George Whaples, A. P. Wheeler, D. V. Widder, V. A. Widder, John Williamson, H. P. Wirth, Jacob Wolfowitz, Alexander Wundheiler, E. H. Zarantonello, Antoni Zygmund.

The meeting opened at 10 A.M. in two sections, one for papers in Applied Mathematics, Professor J. W. Tukey presiding, and one for papers in Analysis, Vice President C. C. MacDuffee presiding. At the general session immediately following these sections, Professor Deane Montgomery gave an address entitled *Transformation groups and spheres*. This address had been originally scheduled for the 1942 Annual Meeting which was cancelled because of transportation difficulties. Vice President J. D. Tamarkin presided at this session.

The afternoon program began at 2 P.M. with a general session at

which Professor Nathan Jacobson gave an address entitled *Some topics in the theory of semi-linear transformations*. At the close of this address, the presiding officer, Vice President C. C. MacDuffee, asked all to rise in memory of two well known mathematicians who had died recently, E. R. Hedrick and David Hilbert.

At 3:15 P.M. there were two sections, one for papers in Algebra and one for papers in Geometry and Statistics. Professors E. R. Lorch and E. J. Moulton were the presiding officers.

The Council met at 12:30 P.M. in the Men's Faculty Club of Columbia University.

The Secretary announced the election of the following fourteen persons to membership in the Society:

Mr. Louis Fein, Earlham College, Earlham, Ind.;
Professor Fred George Fender, Rutgers University;
Mr. N. Knoll Hersh, Tulane University;
Mr. Clarence Thomas Hill, Ponca City, Okla.;
Professor Carl Holtom, Purdue University;
Mr. Morris E. Levenson, Cooper Union, New York, N. Y.;
Mr. Robert Theodore Luginbuhl, University of Pennsylvania;
Professor Glen Treat Miller, Purdue University;
Professor Joseph Edward Morton, Knox College;
Mr. Kenneth Holden Murphy, West Virginia University;
Mr. Frederick Valentine Pohle, College of Engineering, New York University;
Mr. Edward John Specht, Washington University;
Mr. Allen Frederick Strehler, Ohio State University;
Mr. Joseph Simon Vigder, University of Saskatchewan.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

University of California: Messrs. Henry Ludwig Alder, Mark W. Eudey, and Herbert Federer, Misses Evelyn Fix and Mildred Lucille Gerini, Messrs. Eric Leo Lehmann and Robert Russell.
Cornell University: Messrs. Clifford D. Firestone and Alfred Milton Peiser.
Harvard University: Messrs. Richard F. Arens, Robert Creighton Buck, Lindley James Burton, Milton Frank Chauner, Carl Cohen, and James Bruce Crabtree, Miss Ellen Elizabeth Fedder, Messrs. Jaime Lifshitz, Roger Conant Lyndon, and Pesi Rustom Masani, Dr. Eduardo Hector Zarantonello.
Johns Hopkins University: Mrs. Ruth Bari.
University of Kentucky: Miss Elsie Temple Church.
Lehigh University: Messrs. James Levert Howell and Charles Miesse.
Northwestern University: Mr. Richard Lence, Miss Mary Ellen Patno.
Pennsylvania State College: Mr. Israel Harold Rose, Mrs. Alice Margaret Crawford Thorpe.
University of Toronto: Messrs. William James Robert Crosby and Bernard Noonan.
Wayne University: Miss Jane Cronin.
University of Wisconsin: Messrs. Richard V. Andree and Wulf Gotz.
Yale University: Mr. John George Pocock.

The following appointments by the President were reported: as auditors of the Society's books for 1943, Professors Samuel Borofsky and A. E. Meder, Jr.; as Chairman of the Committee on Places of Meetings for the period 1943–1945, Professor W. B. Carver; as a member of the Committee on Places of Meetings for 1943, Professor C. R. Adams; as a member of the Committee on the Award of the Bôcher Prize, Professor Emil Artin; as Committee on Arrangements for the 1943 Summer Meeting at Rutgers University, Professors Richard Morris (Chairman), T. R. Hollcroft, B. W. Jones, A. E. Meder, Jr., C. A. Nelson, and E. P. Starke; as members of the War Policy Committee (joint committee with the Mathematical Association of America), Professors M. H. Stone (Chairman), W. D. Cairns, G. C. Evans, L. M. Graves, Marston Morse, Dr. Warren Weaver, and Professor G. T. Whyburn; as members of a subcommittee of the War Policy Committee, to conduct a bureau of available teachers of mathematics for wartime needs, Professors W. D. Cairns, Arnold Dresden, and J. R. Kline; as Committee on Nomination of Officers and Members of the Council for 1944, Professors E. W. Chittenden (Chairman), E. T. Bell, Saunders MacLane, R. L. Moore, and Dean R. G. D. Richardson.

The Secretary reported that the following had been chosen as the voting representatives of editorial committees on the Council for the year 1943: Bulletin—Dean Tomlinson Fort; Transactions—Professor Einar Hille; Colloquium—Professor G. T. Whyburn; American Journal of Mathematics—Professor Hermann Weyl; Mathematical Reviews—Professor Otto Neugebauer. He also reported that Professor G. T. Whyburn had been selected Chairman of the Colloquium Editorial Committee for 1943.

The Council voted to cancel the meeting scheduled for June 16–17, 1943, at Oregon State College, in connection with the meetings of the Pacific Division of the American Association for the Advancement of Science.

The Council approved September 11–13 as the dates for the 1943 Summer Meeting at Rutgers University.

The Council adopted the following resolution on the death of Professor E. R. Hedrick:

The Council of the American Mathematical Society records its deep sense of loss in the death on February 3, 1943, of Earle Raymond Hedrick. As an active member of the Society during four decades of its unprecedented growth and development, he made contributions which were great in number and varied in character. He gave abundantly of his time, thought, and energy to the Society and served it in the official capacities of Council member, Trustee, Vice President, President, and Editor-in-Chief of the Bulletin. Through membership on many important committees, both

within the Society and outside of it, he labored unceasingly toward the advancement of the interests and prestige of mathematics at all levels. His activities in the Mathematical Association of America, The National Council of Teachers of Mathematics, and in numerous engineering and other scientific societies were extensive and outstanding to a degree hard to comprehend in view of his heavy involvement in Society and other responsibilities. Professor Hedrick had a rare combination of broad interests, outstanding skill at logical and thorough analysis, good judgment, and ability to work effectively with other people. These characteristics invariably singled him out and placed him in a position of leadership.

The Secretary presented a report from President Stone, Chairman of the War Policy Committee, which had its first meeting in Philadelphia on February 21, 1943. Members of the Committee and officers of the Society and Association are maintaining effective contacts with various government agencies, with a view to initiating and accelerating actions which make the best use of mathematically trained and qualified personnel under the Army and the Navy Specialized Training Programs and in research connected with the war. The Committee has recommended that every effort should be made to continue scientific meetings, including those arranged on a national scale.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. Papers numbered 1 to 5 were read in the section for Applied Mathematics, papers 6 to 11 in the section for Analysis, papers 12 to 16 in the section for Algebra, and papers 17 to 22 in the section for Geometry and Statistics; papers 23 to 35 were read by title. Paper 9 was read by Professor Cameron, paper 14 by Dr. Lengyel, paper 19 by Professor Kasner. Mr. Ettlinger was introduced by Professor R. L. Moore, and Dr. Schwerdtfeger by Dr. Peter Scherk.

1. H. J. Greenberg: *Application of a summability method in solving boundary value problems.* (Abstract 49-3-118.)

2. Hilda P. Geiringer: *New convergence cases for iteration methods applied to linear equations.* (Abstract 49-1-62.)

3. Arthur Korn: *On vibrational vortices.* (Abstract 49-1-67.)

4. Stefan Bergman: *A formula for the stream function in compressible fluid flow.* (Abstract 49-3-116.)

5. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.* II. (Abstract 49-5-156.)

6. Wolfgang Wasow: *On a boundary layer problem for a certain linear partial differential equation.* (Abstract 49-1-50.)

7. C. E. Rickart: *Decomposition of additive set functions.* (Abstract 49-1-99.)

8. W. C. Strodt: *Analytic solutions of nonlinear difference equations.* Preliminary report. (Abstract 49-1-45.)

9. R. H. Cameron and W. T. Martin: *An expression for the solution of a class of nonlinear integral equations.* (Abstract 49-3-110.)

10. N. A. Hall: *Confluence of the basic hypergeometric series.* (Abstract 49-1-31.)

11. Raphael Salem: *Sets of uniqueness and sets of multiplicity.* (Abstract 49-3-114.)

12. Gerald Harrison: *The structure of algebraic moduls.* Preliminary report. (Abstract 49-1-4.)

13. Alexander Wundheiler: *An algebraic definition of affine space.* (Abstract 49-1-20.)

14. Saunders MacLane and B. A. Lengyel: *Integral invariants of a tensor under the symmetry operations of the 32 crystal classes.* Preliminary report. (Abstract 49-1-8.)

15. André Weil: *Differentiation in algebraic number-fields.* (Abstract 49-1-19.)

16. Olaf Helmer: *An extension of the elementary divisor theorem.* (Abstract 49-1-5.)

17. Mary E. Ladue: *Conformal geometry of horn angles of higher order.* (Abstract 49-1-74.)

18. Henry Scheffé: *On solutions of the Behrens-Fisher problem, based on the t-distribution.* (Abstract 49-3-121.)

19. Edward Kasner and John DeCicco: *Generalized dynamical trajectories in space.* (Abstract 49-3-120.)

20. W. K. Feller: *On the general form of the so-called law of the iterated logarithm.* (Abstract 49-1-87.)

21. Domina E. Spencer: *The tensor representation of the figures of Study's "Geometrie der Dynamen."* (Abstract 49-1-79.)

22. Jesse Douglas: *Point transformations and isothermal families of curves.* II. (Abstract 49-5-160.)

23. E. F. Beckenbach: *On conjugate harmonic functions.* (Abstract 49-3-109-*t*.)

24. Joseph Bowden: *The quaternary permutation function and a generalization of Newton's binomial theorem and Vandermonde's permutation theorem.* (Abstract 49-3-104-*t*.)

25. Nathaniel Coburn: *Boundary value problems in plane plasticity.* Preliminary report. (Abstract 49-3-117-*t*.)

26. L. L. Dines: *On linear combinations of quadratic forms.* (Abstract 49-3-105-*t*.)

27. M. G. Ettlinger: *On irreducible continuous curves.* (Abstract 49-3-122-*t*.)

28. W. K. Feller: *On a probability limit theorem of H. Cramér.* (Abstract 49-1-86-*t*.)

29. K. O. Friedrichs: *On nonlinear vibrations of third order.* (Abstract 49-1-61-*t*.)

30. O. G. Harrold: *A decomposition theorem for certain compacta.* (Abstract 49-3-123-*t*.)

31. Einar Hille and Gabor Szegö: *Remarks on ergodic theorems.* (Abstract 49-5-138-*t*.)

32. Fritz John: *Instability of certain nonlinear vibrations.* (Abstract 49-1-65-*t*.)

33. H. Schwerdtfeger: *Identities between skew-symmetric matrices.* (Abstract 49-3-106-*t*.)

34. Gabor Szegö: *On the oscillation of differential transforms.* IV. *Jacobi polynomials.* (Abstract 49-3-115-*t*.)

35. J. W. T. Youngs: *On the additivity of the Lebesgue area.* (Abstract 49-1-56-*t*.)

T. R. HOLLCROFT,
Associate Secretary

EARLE RAYMOND HEDRICK—IN MEMORIAM

On the morning of February 3, 1943, the announcement was made from the Rhode Island Hospital at Providence that Dr. Hedrick had just died. He had been a patient with an infected lung since early in November.

When he retired from the office of Provost and the Vice President of the University of California at Los Angeles he accepted a position as Visiting Professor of Mathematics at Brown University, and had just begun his duties there when he was stricken. The shock to the University can well be understood; it was felt throughout the academic world.

It is fitting that we pause to meditate on the life and work of this remarkable man.

Earle Raymond Hedrick was born at Union City, Indiana, September 27, 1876 of Dutch and German ancestry (Hetrig was the earlier spelling). The family migrated to Pennsylvania about 1670.

The principal biographical facts are given adequately in the *Semicentennial history of the American Mathematical Society, 1888–1938, with biographies and bibliographies of the past presidents*, by Professor R. C. Archibald, for 21 years librarian of the Society. This volume contains a complete list of his mathematical writings to that time, and much of the history of his contribution to the development of mathematics in America. The following statements are taken from it.

When the late Frank Nelson Cole retired from his various offices in the Society in 1921 an opportunity was afforded to make numerous fundamental changes in the organization and administration of the Society.

Hedrick was appointed editor-in-chief of the BULLETIN. This office he held until his appointment to that of Provost and Vice President of the University of California at Los Angeles in 1937. It was a period of unparalleled mathematical development in America, and with it were evolved many details in arrangement, style of type, etc., that have contributed in no small degree to the usefulness of the periodical, such as the listings of New Publications, and differentiating in the contents of odd and even numbers. In this process the size of each volume was practically doubled.

Simultaneously with this development in the make-up of the BULLETIN the standard for the acceptance of material became necessarily high, and the effective presentation of the complex material notable. All this required an enormous amount of personal attention

to detail. The total effects of these efforts on the place and the tone of mathematics in America can hardly be overestimated.

In appreciation of this service the Council of the American Mathematical Society directed that volume 44 (1938) of the BULLETIN be dedicated to him. The frontispiece of the volume bears his portrait.

From his earliest years Dr. Hedrick always showed a keen interest in teaching mathematics from the preparatory school to the graduate school. His translation of Goursat's *Cours d'analyse* and (in collaboration with the late C. A. Noble) of the first third of Klein's *Elementarmathematik vom höheren Standpunkte aus* rendered great service to students and college teachers. In 1916 the Mathematical Association of America was founded and Hedrick was elected as its first president.

The effort to improve the quality of teaching was always in the foreground during his professorship at the University of Missouri. A notable contribution is the Series of Mathematical Texts, about forty volumes, of which he was general editor. In addition to secondary school and elementary college texts, this series includes several important works of a more advanced nature.

His interest in mechanics early resulted (1909) in the publication (with the late O. D. Kellogg) of the *Applications of the calculus to mechanics* which proved to be a particularly useful book for those teaching calculus to engineers. Besides his own contributions, he has also been editor of 34 volumes by various authors in the notable Engineering Science Series. Moreover, for five years he was secretary of Section A of A.A.A.S.

His services as administrative officer of the University of California at Los Angeles will be fittingly treated on another occasion.

During the last year he was active in the affairs of the School of Mechanics at Brown University.

In all his numerous and exacting duties, each performed with great patience and consummate skill, the main characteristic of this man was his consideration of human attributes. He was always interested, he was always sympathetic and he was always helpful. Probably no one has put a deeper imprint on the development of numerous lines mentioned here than he.

VIRGIL SNYDER

BOOK REVIEWS

Methods of correlation analysis. By Mordecai Ezekiel. 2d edition. New York, Wiley, 1941. 19+531 pp. $5.00.

The first edition of this well known work appeared in 1930 and had a marked influence upon the users of correlation theory in this country. Before its appearance attention had been centered upon the calculation of correlation coefficients to many decimal places with too much reliance upon probable errors and too much faith in Blakeman's tests. Ezekiel's book emphasized the regression side of correlation with particular stress on nonlinear regression functions. The use of free hand fitting of regression curves in simple and multiple correlation is clearly and quite completely discussed. As a result of the simplicity of explanations and the careful description of calculation procedures, it soon became one of the most important books in the field of correlation.

The difference between the two editions is a survey of the history of the advances in correlation theory during the intervening decade. As stated in the preface of the second edition these major changes have been "first, in the interpretation of the meaning of standard errors and, second in the application of logical limitations to the flexibility of graphic curves. Other significant developments have been in the perfection of new and speedier methods of estimating the reliability of an individual estimate or forecast." Remaining portions of the subject matter are left practically the same.

In general these changes are well made. Treatment of the reliability of an individual forecast is given in Chapter 19. Probability statements arising in the interpretations of standard errors have been correctly made, but the author did not introduce the terminology of "confidence interval" and "fiducial limits." This would be advisable. In discussing logical limitations of graphic curves it is carefully noted that extrapolation is based on these logical considerations rather than the statistical analysis. Samples give most reliable information for the ranges of the variables included in the sample.

Since the general features of the first edition are so well known, it seems most important to mention here some of the detail in the new edition where particular comment is pertinent.

The method of identifying classes by open class limits is superior to the one used. For example (p. 5) change the notation from 22.5–25.4 bushels to 23–25 bushels. The latter form gives the lowest and

highest values included in the class and indicates the accuracy of the measurements.

On p. 10 the text states "Most of the reports come at about the middle values and then thin out to both ends (that is, the distribution approximates normality). In such cases the standard deviation gives a measure of the range within which a definite proportion of the cases will be included." This is followed by a statement of the common area values for the normal curve of error. The author should explain that these area values are only approximate for non-normal distributions which satisfy the given conditions, and that normality requires a particular functional relationship, not merely a graphic similarity.

On p. 156 the following statement appears. "Since the index of determination is simply $\bar{\rho}^2_{yz}$, it is 71.0 per cent." According to the definition given it should be .710.

In another discussion involving the coefficient of determination (p. 159) the following statement is made. "Since the coefficient is a ratio, it is a "pure number," that is, it is an arbitrary mathematical measure, whose values fall within a certain limited range, and it can be compared only with other constants like itself, derived from similar problems." The arithmetic mean is a ratio, and it depends on unit! Furthermore not all ratios are arbitrary! The meaning of the quotation is vague. The point in question is an important one, and a clear explanation should be given.

On pp. 160–162 the author discusses the interpretation of the three types of measure of correlation; regression coefficient, correlation coefficient, and standard error of estimate. He states that the most accurate estimate of values of the dependent variable made from the regression equation calculated from a sample will be made from that sample for which S_y, the standard error of estimate, is the smallest. The accuracy of an estimate of y should be interpreted with reference to the importance of a unit change in y. In a sample with small variation in the dependent variable two situations may be considered: (a) a small variation may be important in case the population itself has small variation, or (b) the dependent variable has been so controlled in sampling that the variation is materially biased more than the usual sampling bias. It is well known that sampling under case (b) tends to lower the correlation coefficient and thus to make the whole correlation analysis unreliable. In case (a) it is clear that S_y must be interpreted in relation to σ_y. The factor $(1-r^2)^{1/2}$ measures the improvement in estimation of y due to the use of regression, and the smaller the value of r, disregarding sign, the smaller is the amount of information supplied by the regression.

In introducing linear multiple regression (p. 164) it should be pointed out that the form of the equation assumes the relationship between the variables to be additive. The discussion given might lead the reader to believe that the linear equation includes all cases.

Statisticians interested in factor analysis will question the recommendation that not more than ten, usually five, variables be used in a study.

To obtain such quantities as $\sum XY$ and $\sum X^2$ it should be mentioned that the extensions as made in the text are not needed. An explanation of the use of a calculating machine to shorten the work might well be given in the text at an early point.

One rather general criticism of the book should be made. There is throughout the tendency to over-correct calculated constants and to over-refine tests of significance. More care should be taken to explain the fact that experimental data frequently do not justify the use of many of these refinements. To an untrained reader they may imply an accuracy of analysis not actually present. In some cases the significance tests suggested are actually incorrect, as for example the use of the standard error of the coefficient of multiple correlation.

In spite of the above criticisms the reviewer considers this book still to be the best in its field.

E. L. WELKER

Finite dimensional vector spaces. By Paul R. Halmos. (Annals of Mathematics Studies, no. 7.) Princeton University Press, 1942. 5+196 pp.

In this book the author presents the topics covered usually in an introductory course in algebra (matrices, linear equations, linear transformations, and so on) from the point of view of a modern analyst interested in general vector spaces.

The ever-growing interest in Hilbert and more general linear spaces makes the appearance of the book very timely, especially since it furnishes an excellent introduction to the subject certainly within the grasp of a first-year graduate student or even a good senior or junior.

The topics are treated in such a manner as to make future generalizations look both natural and suggestive. This sometimes is done at the expense of the shortness of exposition. Some theorems, as the author himself confesses, could be proved in fewer lines. He prefers, however, longer proofs that admit a generalization to shorter ones that do not.

The reviewer finds himself in complete agreement with this method

of presentation, since in the long run it is more economical as far as "economy of thought" is concerned.

Ths book is divided into three chapters: Chapter I, Spaces, Chapter II, Transformations, and Chapter III, Orthogonality. There are also three appendices: on the classical canonical form, on the direct products, and on Hilbert space. This last appendix is just a short preview of what the reader will get into when he takes up further study.

The chapters are divided into sections, and this division was done with great care and didactical skill. The headings of the sections indicate by themselves the impressive variety of topics treated. These range from the relatively familiar ones like linear dependence (§§4, 5), linear manifolds (§9), or proper values of Hermitean and unitary transformations (§62), to those that are of a definitely less familiar and more difficult character like reflexivity of finite dimensional spaces (§15), polar decomposition (§67), or the ergodic theorem for unitary transformations (§76).

However, the reviewer doubts whether this book could be used successfully in a course on matrices and linear equations. The presentation is definitely that of an analyst and the "algebraic" point of view is purposely avoided. The author states in the preface that his purpose is " . . . to emphasize the simple geometric notions common to many parts of mathematics, and to do it in a language which gives away the trade secrets and tells the students what is in the back of the minds of people proving theorems about integral equations and Banach spaces."

The reviewer thinks that the author succeeded admirably in this respect.

MARK KAC

The non-singular cubic surfaces. By B. Segre. Oxford, The Clarendon Press, 1942. 11+180 pp. 63 figures. $4.50.

The extensive literature on the cubic surface would hardly justify further additions unless such additions contributed a novel approach. The novelty of the approach in the case of the volume under review justifies its inclusion.

A non-singular cubic surface F and a triad of planes not belonging to a pencil form a pencil the surfaces of which remain non-singular in all the intermediate positions from F to the triad of planes. Under these circumstances the lines on the surface are always distinct and preserve their incidence relations, thus giving rise to the group.

The book is divided into four chapters devoted to (1) the discussion

of the lines and the configurations formed by them, (2) the group, (3) the properties of the surface under the restriction that it shall be real, with all possibilities of real and imaginary lines, the latter occurring in pairs, (4) the exceptional cases of the pentahedroid.

In the limiting triad of planes, intersecting by pairs in three lines, each of which meets F in three points, the 27 lines are represented by joining these points in pairs. Consider a symbol abc, one of which one is zero, and each of the others may be 1 or 2, or 3, independently of each other. The numerical value and the position of each letter in the symbol has meaning. Consider the points 1, 2, 3 on each line r_1, r_2, r_3 in which two planes of the triad intersect. The position of a refers to points on r_1, and so on. Thus, the symbol 012 is the (image) line having no point on r_1, having point 1 on line r_2 and the point 2 on r_3. Two lines abc and $a'b'c'$ intersect when and only when these symbols have one and only one element as $a=a'$, or $b=b'$, or $c=c'$ alike. The configuration of the lines of F can now be expressed in terms of simple combinatorial relations in these symbols and expressed graphically in axonometric projection. Thus, the following theorems are samples. They are immediate consequences of the symbolism.

Every line meets ten others.

Given a line r, and any two others, skew to it and to each other; then three other lines can be found having similar relations among themselves, but each new line meeting all the lines of the first group.

Given any line r, there are five others skew to it and to each other.

The properties of the double sixes and of the tritangent planes follow at once. A plane through a line r meets F in a conic which meets r in two points of an involution, the double points of which on the three lines of any tangent plane lie by threes on the sides of a complete quadrilateral.

The second chapter discusses the group of the 27 lines from the property that incident relations are preserved. It is shown to have six independent generators, directly suggested from the graphical representation. It contains a conjugate set of 36 involutorial operations, called σ-transformations, each of which interchanges the conjugate lines of two complementary sextuples and leaves unaltered the remaining 15. The 15 pairs of parabolic points of the 15 lines of F residual to a double six are conjugate with respect to a Schur quadric of the double six. The six pairs of parabolic points of the lines of any two complementary triplets lie on a space quartic curve of the first kind.

The third chapter—on real non-singular cubic surfaces—is the main theme discussed in the book; it contains about two-thirds of the entire volume. Consider the pencil $F+\lambda\phi=0$, wherein ϕ is a nodal

cubic. The six lines of F through the node replace the 12 lines of a double six; the incidence relations of the remaining 15 are those of F. The classification of surfaces F into four distinct types, according to the number of pairs of conjugate imaginary lines, and the addition of one more type with 12 pairs of conjugate imaginary lines, now proceeds as in the Schläfli enumeration, supplemented by topological consideration of the space [19] of all cubic surfaces and the primal representing the discriminant of the surface. It is shown that these five types comprise all possible forms of non-singular real cubic surfaces. Each of these types is then considered in more detail, the graphical representation being applied to each, with special devices for representing the imaginary lines of the first and second kind. Expressed in this way, the configurations become obvious; the various kinds of double sixes are immediately apparent. The groups of the various types are considered further, featuring the differences between right-hand and left-hand lines as determined by the sequence of parabolic planes and the planes of composite residual section, as the plane turns about the line. This study is then applied to the configurations of the real sextuples of lines of each type, and finally to the topology of real cubic surfaces in connection with the real lines. With this preparation the form of the parabolic curve on each type is determined.

The last chapter is analytic. It is devoted to the Sylvester pentahedron and to those cubic surfaces invariant under homographic transformations. The general cubic surfaces can be expressed by an equation involving five cubes of linear forms in one and only one way. The only non-singular cubic surfaces for which there is a point V such that all their sections by planes through V have the same modulus are the cyclic surfaces having V as center. Conversely, all the plane sections of the non-conical cyclic cubic surfaces are equianharmonic. A real cubic surface with 27 distinct real lines has always a well determined Sylvester pentahedron, consisting of 5 distinct real planes, no four of which are dependent.

Three short appendices appear at the end of the book, devoted respectively to the degeneration of a cubic surface into a plane and a quadric, the degeneration of a non-singular cubic primal of [4] into three primes, and on singular surfaces of a pencil of cubic surfaces.

The press-work is of the expected excellence of the Clarendon Press. The 63 figures are particularly good. The volume is strikingly free of typographical errors.

VIRGIL SNYDER

NOTES

A copy of G. N. Watson's *Theory of Bessel's Functions* is urgently needed in connection with important war research. Anyone who has a copy he would be willing to sell, or to lend for the duration, is requested to communicate with Dr. Warren Weaver, Chief of the Applied Mathematics Panel, National Defense Research Committee, Room 5500, 49 West 49th Street, New York, N. Y.

Nature reports a tercentenary celebration of Isaac Newton in Great Britain in January. Professor Max Born of the University of Edinburgh was chairman of the celebration and the program included an address by Professor E. T. Whittaker.

The Association for Symbolic Logic has announced the election of the following officers for the year 1943: President, Professor C. H. Langford; Vice President, Professor C. A. Baylis; Secretary-Treasurer, Miss Helen C. Brodie; Members of Executive Committee, Professor Arnold Dresden and Professor F. B. Fitch; Member of Council, Dr. A. M. Turing.

The 1942 Lamme Medal of the American Institute of Electrical Engineers has been awarded to Dr. Joseph Slepian of the Westinghouse Electric and Manufacturing Company.

Dr. H. G. Barone has been appointed to an assistant professorship at Pennsylvania State College.

Dr. H. F. Bright has been appointed to an assistant professorship at Denison University.

Professor Emeritus W. D. Cairns of Oberlin College has been appointed visiting professor of mathematics at the University of New Mexico.

Dr. Nancy Cole of Sweet Briar College is visiting assistant professor at Kenyon College.

Dr. D. W. Hall of Brown University has been appointed to an assistant professorship at the University of Maryland.

Dr. J. F. Heyda of Michigan State College has been appointed to an assistant professorship at Denison University.

Associate Professor R. C. Hildner of Mt. Union College has been appointed to an assistant professorship at the College of Wooster.

Assistant Professor R. N. Johanson of Bradley Polytechnic Institute has been appointed to an assistant professorship at Hamilton College.

Assistant Professor E. M. Justin of the Case School of Applied Science is on leave of absence and is at the University of California.

Dr. Joseph Lehner of Cornell University has accepted a position with the Kellex Corporation of New York City.

Professor E. J. Moulton of Northwestern University is on leave of absence, and Professor H. T. Davis is acting head of the department of mathematics.

Assistant Professor W. H. Myers of San Jose State College has been appointed acting head of the department of mathematics.

Mr. N. D. Nelson of the University of Wisconsin has been appointed to an assistant professorship at Amherst College.

Dr. E. A. Nordhaus of the University of Wisconsin at Milwaukee has been appointed to an assistant professorship at Michigan State College.

The following appointments to instructorships are announced: Allegheny College: Mr. R. E. Smith; Brown University: Mr. D. A. Jonah, Mr. P. C. Rosenbloom; University of Chicago: Mr. H. L. Meyer: Cornell University: Mr. F. W. Dittman; Denison University: Mr. Paul Cramer, Mr. A. C. Moeller, Mr. Robert Sutton, Miss Elaine Van Aken, Mr. Z. A. Wilson; Hamilton College: Dr. V. O. McBrien, Mr. Fritz Steinhardt; Illinois Institute of Technology: Dr. Edward Helly; University of North Carolina: Mrs. F. D. Jacobson, Mr. J. G. McAllister, Mr. E. P. Pearce, Mr. C. A. Phillips, Dr. C. M. Smith, Mr. C. B. Toxey; Pennsylvania State College: Dr. Helen B. Owens, Mr. G. B. Snyder, Mrs. Alice C. Thorpe; University of Rochester: Mr. Frederick Bagemihl; Rutgers University: Mr. J. N. B. Livingood.

The death of Professor David Hilbert of Göttingen University has been reported. He was eighty-one years old.

Dr. W. A. Granville died February 4, 1943, at the age of 79 years. He had been a member of the Society from 1897–1934.

Miss Lilian Hackney of Marshall College died February 4, 1943. She had been a member of the Society since 1923.

Professor H. F. Minssen, chairman of the department of mathe-

matics and vice president of San Jose State College, died February 7, 1943.

Professor H. H. Mitchell of the University of Pennsylvania died March 13, 1943. He had been a member of the Society since 1909.

Professor Emeritus L. M. Passano of Massachusetts Institute of Technology died January 30, 1943.

Assistant Professor N. S. Risley of Fenn College died December 30, 1942.

The death of Mr. Oscar Schmiedel has been reported. He had been a member of the Society since 1892.

Professor M. H. Tyler of Rhode Island State College died December 16, 1942.

Professor J. J. Westemeier of Dowling College, Des Moines, Iowa, died February 1, 1943.

The following seventy-five doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1942 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

Claire F. Adler, New York University, June, minor in physics, *An isoperimetric problem with an inequality.*

B. H. Arnold, Princeton, June, *Rings of operators on vector spaces.*

S. P. Avann, California Institute of Technology, June, minor in physics, I. *Theory of the τ- and σ-functions in semi-modular lattices.* II. *Lattice automorphisms.*

Sister Saint Augustine Ball, Catholic, November, minor in education, *Power functions and distribution functions related to "Student's test."*

J. C. Bell, Illinois, October, minors in astronomy and physics, *An extension of the parallelism of Cliffird.*

T. A. Botts, Virginia, June, *Sufficient conditions for generalized-curve problems in the calculus of variations.*

H. K. Brown, Michigan, May, *The resolution of boundary value problems by means of the finite Fourier transformation.*

H. E. Burns, Northwestern, June, *Linear operators expressed as Green's functions with special application to the elastic plate problem.*

W. Z. Chien, Toronto, December, minor in civil engineering, *The intrinsic theory of elastic shells and plates.*

D. E. Christie, Princeton, June, *Net homotopy for compacta.*

Paul Civin, Duke, May, minor in physics, *Two-to-one mappings of manifolds.*

W. G. Clark, Kentucky, June, *On generalized quaternion algebras and universal Hermitian forms.*

I. S. Cohen, Johns Hopkins, May, *The structure and ideal theory of local rings.*

L. L. Cronvich, Wisconsin, June, *Deformation of naturally twisted compound beams.*

W. F. Eberlein, Harvard, June, *Closure, convexity, and linearity in Banach spaces.*

W. C. G. Fraser, Toronto, June, *The factorial transform.*

Louis Garfin, Iowa, May, *A comparative study of the underlying principles of certain pension schemes for a staff of employees with special reference to teachers and public employees.*

T. S. George, Duke, May, minor in physics, *Concerning the equilibrium point of the Green's function for an n-dimensional spherical annulus.*

H. F. Gingerich, Illinois, June, minor in physics, *Generalized fields and Desargues configurations.*

Caspar Goffman, Ohio State, June, *On the converses of certain theorems on the symmetric structure of sets and functions.*

Saul Gorn, Columbia, May, *Homomorphisms and modular functionals.*

R. W. Hamming, Illinois, June, minor in physics, *Some problems in the boundary value theory of linear differential equations.*

C. A. Hayes, California (Berkeley), May, *Boundary value problems for elliptic systems of linear partial differential equations.*

L. B. Hedge, Brown, May, I. *Moment problem for a bounded region.* II. *Transformations of multiple Fourier series.*

R. G. Helsel, Ohio State, August, *A geometrical application of integral means.*

Edwin Hewitt, Harvard, June, *A problem of set-theoretic topology.*

Carl Holtom, Chicago, September, *Permanent configurations in the n-body problem.*

Robert Hooke, Princeton, January, *Linear p-adic groups and their Lie algebras.*

R. P. Isaacs, Columbia, June, *A finite difference function theory.*

G. K. Kalisch, Chicago, June, *On special Jordan algebras.*

Samuel Kaplan, Michigan, May, *Homologies in metric separable spaces.*

William Karush, Chicago, June, *Isoperimetric problems and index theorems in the calculus of variations.*

P. J. Kelly, Wisconsin, June, *On isometric transformations.*

D. E. Kibbey, Illinois, February, minor in physics, *Boundary values of analytic functions.*

Mary E. Ladue, Columbia, June, *The conformal geometry of horn angles of higher order.*

P. A. Lagerstrom, Princeton, June, *Measure and integral in partially ordered spaces.*

J. A. Larrivee, Catholic, May, minor in physics, *Effect of the singularities of a rational curve on the symmetric correspondence determined by residuals on tangents.*

Howard Levi, Columbia, May, *On the structure of differential polynomials and on their theory of ideals.*

J. V. Lewis, California (Berkeley), October, *The existence of solutions to Lagrange problems for multiple integrals.*

L. H. Loomis, Harvard, February, *Some studies on simply-connected Riemann surfaces*: I. *The problem of imbedding.* II. *Mapping on the boundary for two classes of surfaces.*

R. R. R. Luckey, Cornell, June, *Certain applications of Fourier integrals.*

V. O. McBrien, Catholic, May, minor in physics, *On some systems of orthopolar and Kantor lines related to a quadrangle.*

G. W. Mackey, Harvard, June, *The subspaces of the conjugate of an abstract linear space.*

Murray Mannos, Harvard, June, *Vector lattices of finite dimension.*

N. S. Mendelsohn, Toronto, June, *A group-theoretic characterization of the general projective collineation group.*

C. F. Meyer, George Washington, June, *The theory of the motion and capture of free electrons in crystal dielectric media.*

Josephine M. Mitchell, Bryn Mawr, June, *On double Sturm-Liouville series.*

Vladimir Morkovin, Wisconsin, June, *On the deflection of anisotropic thin plates.*

F. G. Myers, Virginia, June, *On sufficiency conditions for the problem of Lagrange.*

Anne F. O'Neill, Radcliffe, June, *Contributions to the theory of derivates.*

C. R. Phelps, Harvard. June, *A homogeneous algebra with limited associativity.*

E. J. Pinney, California Institute of Technology, June, minor in physics, *Calculus of variations in abstract spaces and related topics.*

Harry Pollard, Harvard, February, *Studies on the Stieltjes transform.*

A. L. Putnam, Harvard, June, *Integral domains complete in a valuation.*

S. E. Rauch, Stanford, June, minor in physics, *Mapping properties of the second arithmetic mean of the geometric series.*

D. L. Robb, Catholic, May, minor in physics, *Special cases of the symmetric correspondence C_6 and their relations to the general C_6.*

Sister Mary de Pazzi Rochford, Notre Dame, May, *Integrability conditions without differentiability assumptions.*

J. D. Rommel, Kentucky, August, *On conservative transformations of functions of two variables.*

R. D. Schafer, Chicago, June, *Alternative algebras over an arbitrary field.*

Robert Schatten, Columbia, December, *On the direct product of Banach spaces.*

C. W. Seekins, California Institute of Technology, June, minor in physics, *Lattice multiplications and residuations.*

H. L. Shih, Harvard, February, *Mappings of a 2-manifold into a space.*

R. D. Specht, Wisconsin, June, *Secondary effects in the simple bending of elliptical beams.*

Domina E. Spencer, Massachusetts Institute of Technology, June, minor in electrical engineering, *A tensor interpretation of Study's "Geometrie der Dynamen."*

Zenon Szatrowski, Northwestern, September, minor in economics, *Solutions of some integro-differential equations.*

H. S. Thurston, Wisconsin, June, *The P-adic numbers of Hensel.*

R. H. Tripp, Iowa State, March, minors in physics and theoretical and applied mechanics, *Statical equilibrium of skew and sector shaped plates.*

Alice E. Turner, Chicago, September, *Singularities of space curves.*

Bernard Vinograde, Michigan, May, *Split rings and their representation theory.*

G. L. Walker, Cornell, February, *Direct product and Lorentz matrices.*

W. R. Wasow, New York University, June, minor in physics, *On boundary layer problems in the theory of ordinary differential equations.*

P. A. White, Virginia, June, *r-regular convergence spaces.*

J. E. Wilkins, Chicago, December, *Multiple integral problems in parametric form in the calculus of variations.*

Jacob Wolfowitz, New York University, June, minor in physics, *Additive partition functions and a class of statistical hypotheses.*

G. S. Young, Texas, June, *Concerning the outer boundaries of certain connected domains.*

The following doctorates were conferred in 1941, but were not in-

cluded in the list in the preceding volume of this BULLETIN (vol. 48, pp. 346–352).

J. A. Daum, Nebraska, August, minor in physics, *Basic hypergeometric series.*

C. H. Frick, North Carolina, June, *The development of formulas for columns with varying moment of inertia and end conditions.*

D. C. May, Princeton, October, *An integral formula for analytic functions of k variables with some applications.*

A. H. Stone, Princeton, October, *Connectedness and coherence.*

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of the issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

125. A. T. Brauer: *On the non-existence of odd perfect numbers of form* $p^{\alpha}q_1^2q_2^2\cdots q_{t-1}^2q_t^4$.

If an odd perfect number exists, it must have the form $n=p^{\alpha}q_1^{2\beta_1}q_2^{2\beta_2}\cdots q_t^{2\beta_t}$ where $p, q_1, q_2, \cdots, q_t$ are primes and $p\equiv\alpha\equiv 1 \pmod 4$. This was proved by Euler. Sylvester obtained estimates for t, in particular $t\geqq 4$, and $t\geqq 7$ if $n\not\equiv 0 \pmod 3$. Recently, it was shown by R. Steuerwald (Sitzungsberichte der Bayerischen Akademie der Wissenschaften 1937) that the case $\beta_1=\beta_2=\cdots=\beta_t=1$ is impossible, and by H. J. Kanold (J. Reine Angew. Math. vol. 183 (1941)) that the same is true for $\beta_1=\beta_2=\cdots=\beta_t=2$. Moreover Kanold proved that n is not perfect if the greatest common divisor d of $2\beta_1+1, 2\beta_2+1, \cdots, 2\beta_t+1$ is divisible by 9, 15, 21, or 33, and some similar results. All these results deal with the case $d>1$. In this paper it is proved that no odd number of form $p^{\alpha}q_1^2q_2^2\cdots q_{t-1}^2q_t^4$ exists; here $d=1$. For the proof some theorems of T. Nagell on Diophantine equations are used (Norsk Matematisk Forenings Skrifter 1921). (Received March 24, 1943.)

126. R. P. Dilworth: *Lattices with unique complements.*

For some time an outstanding problem in lattice theory has been the following: Is every lattice having unique complements a Boolean algebra? It is shown here that the statement is *not* true. Indeed the following counter theorem is proved: Every lattice is a sublattice of a lattice with unique complements. (Received March 20, 1943.)

127. J. E. Eaton: *A Galois theory for differential fields.*

Let Λ be an algebraically transcendental extension of the partial differential field Γ, and consider all isomorphisms of Λ leaving Γ invariant. These isomorphisms may be grouped into a finite number of disjoint sets $\{m_i\}=\mathfrak{M}$ which, with a suitably defined multiplication, is a multigroup called the Galois multigroup of Λ over Γ. There is a 1-1 correspondence between the subfields of Λ containing Γ and the submultigroups of $\mathfrak{M}$ such that if $\Delta\rightleftarrows\mathfrak{H}$, then $\mathfrak{H}$ is the Galois multigroup of Λ over Δ and $\mathfrak{M}//\mathfrak{H}$ (the multiplicative system of the double coset decomposition of $\mathfrak{M}$ with respect to $\mathfrak{H}$) is the Galois multigroup of Δ over Γ. These results are an extension of the work of H. W. Raudenbush (*Hypertranscendental extensions of partial differential fields*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 714–720) and E. R. Kolchin (*Extensions of differential fields*, I, Ann. of Math. vol. 43 (1942) pp. 724–729). (Received March 25, 1943.)

128. D. H. Lehmer: *On Ramanujan's numerical function $\tau(n)$.*

Ramanujan's numerical function $\tau(n)$ may be defined by $\sum\tau(n)x^{n-1}=\{\pi(1-x)^k\}^{24}$. Among several unsolved questions about $\tau(n)$ is the so-called Ramanujan hypothesis to the effect that if p is a prime $\left|\tau(p)\cdot p^{-11/2}\right|<2$, which Ramanujan verified for primes $p<30$. The present writer, in attempting to disprove this important hypothesis, has examined all primes $p<300$, as well as $p=571$, and finds that in all these 47 cases the hypothesis holds true. It nearly fails for $p=103$ when $\tau(103)\cdot 103^{-11/2}=-1.918\cdots$. In connection with this hypothesis Rankin proved in 1939 that as $x\to\infty$, $x^{-12}\sum_{n\leqq x}\tau(n)^2$ tends to a limit represented by a certain double integral extended over the fundamental region of the full modular group. In this paper a practical method is devised for evaluating this integral whose value is found to be .0320047918814 $\cdots$. Various congruence and divisibility properties of $\tau(n)$ are also discussed. For example $\tau(n)$ is composite for $1<n<7921$. (Received March 26, 1943.)

129. A. E. Ross: *Positive quaternary quadratic forms representing all integers with at most k exceptions.*

In this paper it is shown that there are a finite number of classes of positive quaternary quadratic forms which represent all integers with at most k exceptions. The determinants of such forms have an upper bound B_k depending on k. This is a generalization of the results of Ramanujan (Proc. Cambridge Philos. Soc. vol. 19 (1917) pp. 11–21), Ross (Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) p. 607) and Halmos (Bull. Amer. Math. Soc. vol. 44 (1938) pp. 141–144). Ross gives $B_0=112$ for the classic case and Halmos' results imply that $B_1\geqq 240$ in the classic case. (Received March 26, 1943.)

130. L. I. Wade (National Research Fellow): *Transcendence properties of the Carlitz ψ-function.*

The paper is concerned with quantities transcendental over the field $GF(p^n, x)$. For the Carlitz ψ-function (L. Carlitz, Duke Math. J. vol. 1 (1935) pp. 137–168) and its inverse $\lambda(t)$ the following theorem is proved. If β is algebraic (over $GF(p^n, x)$) and irrational and if $\alpha\neq 0$ is algebraic, then $\psi(\beta\lambda(\alpha))$ is transcendental over $GF(p^n, x)$. In a sense this is an analogue of Hilbert's seventh problem for the transcendence of $\alpha^\beta=e^{\beta\log\alpha}$ over the rational number field. (Received March 26, 1943.)

Analysis

131. L. W. Cohen: *On linear equations in Hilbert space.*

If the rows of the infinite matrix $A=\|a_{ij}\|$ are points in Hilbert space and $a^{i_1\cdots i_m}_{j_1\cdots j_m}$ are the m-rowed determinants with elements in A, it is shown that $\det A_{i_1\cdots i_m}B'_{i_1\cdots i_m}=\sum_{i_1\cdots i_m}a^{i_1\cdots i_m}_{j_1\cdots j_m}b^{i_1\cdots i_m}_{j_1\cdots j_m}$ where $A_{i_1\cdots i_m}$, $B_{i_1\cdots i_m}$ are m-rowed minors of A, B respectively. The series is summed over all combinations of integers $j_1, \cdots, j_m$ and converges absolutely. This identity is used to establish sufficient conditions in order that the linear system represented by $Ax=y$ have a solution in Hilbert space for all y in that space. (Received March 24, 1943.)

132. R. J. Duffin: *Some representations for Fourier transforms.*

Let $\phi(x)$ be an arbitrary function and let $f(x)$ and $g(x)$ be defined by the series: $f(cx)=\sum_1^\infty(-1)^n\phi((2n-1)/x)/x$, $g(cx)=\sum_1^\infty(-1)^n\phi(x/(2n-1))/(2n-1)$. Then if

$c=\pi^{1/2}/2^{1/2}$, $g(x)$ is the Fourier sine transform of $f(x)$. The purpose of this paper is to give several conditions for the validity of these formulas. (Received March 26, 1943.)

133. F. A. Ficken: *Note on the existence of scalar products in normed linear spaces.*

In order that a normed linear space Σ with complex scalars may permit the definition of a scalar product, it is necessary and sufficient that, whenever a and b are real scalars and A and B are vectors, $|A|=|B|$ shall imply (α): $|aA+bB|=|bA+aB|$. The necessity of this condition is easily verified. To establish its sufficiency (α) is used to prove, for any vectors A and B, that (β): $|A+B|^2+|A-B|^2=2|A|^2+2|B|^2$. The sufficiency (and necessity) of condition (β) for the existence of a scalar product was established by J. von Neumann and P. Jordan (Ann. of Math. vol. 36 (1935) pp. 719–723). (Received March 29, 1943.)

134. F. A. Ficken: *On two criteria for convexity in a symmetric Banach space.*

The results refer to a Banach space Σ with real scalars and a scalar product. It has been shown by Moskovitz and Dines (Duke Math. J. vol. 5 (1939) pp. 520–534; Bull. Amer. Math. Soc. vol. 46 (1940) pp. 482–489) that: If a set E is closed and has interior points, then E is convex if and only if E has a supporting plane at each point of a set which is everywhere dense in its frontier E'. In their work a supporting plane is required to contain a point of E, and hence can exist only at points of E' which belong to E. Dropping this requirement, their results are used to obtain slightly stronger conclusions without assuming E to be closed. Another criterion, established by Jessen for Euclidean n-space (cf. Math. Rev. vol. 2 (1941), p. 261) requires the presence, for each point of the complement of E, of a unique nearest point on the closure of E. By generalizing the familiar geometric transformation of inversion, this criterion is established for the convexity of a compact set, or of any set if Σ has the property that each bounded subset is compact. (Received March 29, 1943.)

135. P. R. Halmos: *Approximation theories for measure preserving transformations.*

Three topologies (called neighborhood, uniform, and metric) are introduced into the set G of all (1, 1) measure preserving transformations of a finite measure space onto itself. The first two of these are the analogues of the strong and the uniform topologies, respectively, of the set of bounded operators on Hilbert space and the third is defined by the distance function $d(S, T)=m\{x \mid Sx \neq Tx\}$. It is proved that the uniform and the metric topologies are identical; in both the neighborhood and the metric topologies G becomes a complete topological group. The central results in both cases assert that arbitrary measure preserving transformations may be approximated by transformations with comparatively simple properties (for example, by transformations of finite period). It follows from these approximation theorems that the adage according to which "in general a measure preserving transformation is ergodic" is true in the neighborhood topology (that is, the set of ergodic transformations is a residual G_δ) and false in the metric topology (that is, the set of ergodic transformations is nowhere dense). (Received March 2, 1943.)

136. R. W. Hamming: *The asymptotic location of the roots of exponential sums having polynomial exponents.*

This paper considers the asymptotic location of the zeros of $f(z)=\sum_{i=1}^{n}P_i(z)\exp Q_i(z)$, where $P_i(z)$ and $Q_i(z)$ are polynomials in z. An inequality concerning certain regions covering the zeros is also given. (Received March 1, 1943.)

137. R. G. Helsel and Tibor Radó: *The transformation of double integrals.*

Radó and Reichelderfer have established, for certain wide classes of continuous transformations in the plane, extremely general topological transformation formulas for double integrals. The purpose of the present paper is to show that their results imply all of the results on the transformation of double integrals in the literature of which the authors are aware. Particular attention is given to the work that W. H. Young has done on the transformation of double integrals. The paper also contains new results on approximation by integral means. (Received February 19, 1943.)

138. Einar Hille and Gabor Szegö: *Remarks on ergodic theorems.*

Ergodic theorems are considered from the point of view of classical theory of summability. In particular, Abelian and Tauberian theorems give relations between the existence of (A)-lim T^n and (C, k)-lim T^n in a given topology. A detailed study is made of the transformation $T=T_\alpha=I-U_\alpha$ where $U_\alpha f$ is the fractional integral of f of order α. Here the resolvent has an isolated essential singular point at $\lambda=1$, but (A)-lim $T_\alpha^n=0$ for $0<\alpha<2$ in the strong (but not in the uniform) topology of $L_1(0, 1)$ as well as almost everywhere. For $0<\alpha<1$ even lim $T_\alpha^n=0$ and for $\alpha=1$, (C, k)-lim $T_1^n=0$ for $k>1/2$. (Received February 27, 1943.)

139. H. K. Hughes: *On a theorem of Newsom.*

A theorem proved by Newsom (Amer. J. Math. vol. 60 (1938) pp. 561–572) furnishes an expression from which it is often possible to obtain the asymptotic expansions for large values of $|z|$ of the integral function $f(z)=\sum_{n=0}^{\infty}g(n)z^n$ (radius of convergence $=\infty$), the series being regarded as given. It is assumed that the function $g(w)$, where $w=x+iy$, is single-valued and analytic in the finite w plane, and is less in absolute value than a constant multiple of $e^{k\pi|y|}$, k being a positive integer. The present paper considers the situation when $g(w)$ has a singularity in the finite plane, while the inequality remains satisfied for sufficiently large values of $|w|$. The result is that a certain loop integral must be subtracted from the expression obtained by Newsom. A method of obtaining the asymptotic expansion of the loop integral is indicated. (Received March 25, 1943.)

140. H. D. Huskey: *Contributions to the problem of Geöcze.*

Consider a continuous surface $S: z=f(x, y)$ where $f(x, y)$ is continuous on the unit square I: $0\leq x\leq 1$, $0\leq y\leq 1$. The area of such a surface is given by $A(S)$ = greatest lower bound lim inf $A(P_n)$ where P_n: $z=p_n(x, y)$, $p_n(x, y)\rightarrow f(x, y)$ uniformly on I, and the greatest lower bound is taken with respect to all such sequences of polyhedra (not necessarily inscribed). Let $A^*(S)$ denote the value obtained in the preceding definition if the polyhedra are required to be inscribed. The problem of Geöcze asks: When is $A(S)=A^*(S)$? The author has used integral means and results on convergence in area (Radó and Reichelderfer, *Convergence in length and convergence in area*, Duke Math. J. vol. 9 (1942) pp. 527–565) to obtain two different proofs that $A(S)=A^*(S)$ if $f(x, y)$ is absolutely continuous in the sense of Tonelli. (Received February 19, 1943.)

141. D. H. Hyers: *A note on the Michal differential in Banach spaces.*

In this note an example is given which shows that the Michal M-differential (Proc. Nat. Acad. Sci. U.S.A. vol. 24 (1938) pp. 340–342 and Bull. Amer. Math. Soc. vol. 45 (1939) pp. 529–563) is more general than the Fréchet differential in Banach spaces. (Received March 27, 1943.)

142. R. E. Lane: *The values of continued fractions and their computation.*

This paper gives a method of computing the differences between the values of successive approximants to a continued fraction and shows how it is often possible to find the maximum possible absolute value of the difference between the value of the nth approximant and the value of the continued fraction. The principal result, the two-circle theorem, gives a necessary and sufficient condition on s for a certain linear fractional transformation, $v=t_s(u)$, to carry $|u-c| \leqq |c|$ into some portion of $|v-d| \leqq |d|$. The Worpitzky circle, the Paydon-Scott-Wall parabolas, and various other element-regions for continued fractions arise as special cases. Among the other applications of the two-circle theorem is a simple proof that if the elements of a continued fraction lie in a certain parabola, the approximants lie in the inner loop of a certain limacon. A generalization of this result is given. (Received March 27, 1943.)

143. L. H. Loomis: *Spherical volume as a coordinate-free basis for Lebesgue measure theory.*

The formula for the volume of a sphere is deduced in metric spaces having certain homogeneity properties in common with Euclidean spaces. The fundamental properties of spherical volume are established, and the whole of classical measure theory which does not concern a specific dimension number is then immediately available. The development of Lebesgue measure theory is thus freed from the background of a coordinate representation as the nth power (in the sense of Cartesian product) of the real number system. (Received March 15, 1943.)

144. Glynn Owens: *A boundary value problem for an ordinary nonlinear differential equation of the second order.*

The uniqueness and existence in the infinite region $I[0 \leqq x < \infty]$ of the solution of the boundary value problem (A): $d^2y/dx^2 - \lambda f(x, y)y = q(x, y)$, $y(0) = 0$, $\lim_{x\to\infty} y(x) = 0$, is to be demonstrated in this paper. A fundamental assumption will be that $f(x,y)$ is positively and uniformly bounded away from zero, and it is necessary to assume that the parameter λ is positive and sufficiently great. In the first section of the paper the uniqueness and existence of the solution of a certain linear equation is considered. In addition certain bounds for the solution are derived which are valid respectively in the region I and in a suitable neighborhood of infinity. The principal result, namely the treatment of problem (A), is dealt with in section two. There, by linearizing the problem (A), subjecting the coefficients of (A) to certain conditions, such as Lipschitz requirements and the existence of certain integrals taken over the region I, it is proved, with the aid of the results of section one, that the method of successive approximations used gives a sequence of functions which converge uniformly in I to the desired solution of (A). (Received March 24, 1943.)

145. George Pólya: *Inequalities for the area of the ellipsoid.*

Let $E=E(a, b, c)$ denote the area of the ellipsoid with semiaxes a, b, c. Let λ, μ, ν satisfy the conditions $\lambda \geqq \mu \geqq \nu \geqq 0, \lambda+\mu+\nu=2$. Define $P_{\lambda\mu\nu}=P_{\lambda\mu\nu}(a, b, c)=4\pi\sum a^{\lambda}b^{\mu}c^{\nu}/6$; the sum has 6 terms, obtained from $a^{\lambda}b^{\mu}c^{\nu}$ by performing all possible permutations of a, b, c. Consider linear combinations $P=kP_{\lambda\mu\nu}+k'P_{\lambda'\mu'\nu'}+\cdots$, assume that $k>0, k'>0, \cdots$ and $k+k'+\cdots=1$; the number of the terms is finite, $\geqq 1$. Now P is just as E, a continuous function of a, b, c for $a \geqq 0, b \geqq 0, c \geqq 0$, positive for $a>0$, $b>0, c>0$, homogeneous of degree 2, symmetric in a, b, c, and it has the same value as E for $a=b=c$ (for spheres). The problem is to find all linear combinations P "comparable" with E (see G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, p. 5). A complete solution is given an essential part of which can be stated thus: The inequalities $E>P_{1,1,0}$, $E<(P_{2,0,0}+P_{1,1,0})/2$ hold for $a>0, b>0, c>0$ and are the best of their kind. The linear combination $(64P_{1,1,0}-2P_{2,0,0}-27P_{2/3,2/3,2/3})/35$ is not a P, because it has negative coefficients; but it gives a remarkably good numerical approximation to E if the ellipsoid is not very different from a sphere. (Received March 26, 1943.)

146. Robert Schatten: *On certain properties of crossnorms.*

For two Banach spaces E_1, E_2, a Banach space $E_1 \otimes E_2$ is constructed with a crossnorm N. With $E_1 \otimes E_2$, there is considered the associate Banach space $\overline{E}_1 \otimes \overline{E}_2$ with a norm $\overline{N}$, associate with N. Similarly $\overline{N}$ is defined in $\overline{E}_1 \otimes \overline{E}_2$. N is minimal if $\overline{N}^*=\overline{N}$ implies $N^* \geqq N$. N is reflexive if $\overline{\overline{N}}=N$. N has an "associate property" if for a certain N^*, $\overline{N}^*=N$. It is shown that minimal and reflexive norms and those having an associate property are identical. The existence of a least crossnorm is proved, and that the associate with every crossnorm is also a crossnorm. It is also shown that a uniformly convex crossnorm sets up the relation $\overline{E_1 \otimes E_2}=\overline{E}_1 \otimes \overline{E}_2$, if and only if it is minimal. Let k denote a natural number. The values of a crossnorm for all expressions of rank at most k do not necessarily determine the crossnorm. Finally "semi-self-associate" crossnorms are constructed. In particular, when E_1, E_2 denote two Hilbert spaces, for every natural k, minimal crossnorms S_k are constructed, such that $S_k \neq S$, and $S_k=\overline{S}_k=S$ for all expressions of rank at most k, where S denotes the self-associate crossnorm for Hilbert spaces constructed by F. J. Murray and J. von Neumann (Ann. of Math. (2) vol. 37 (1936) pp. 118–125). (Received February 16, 1943.)

147. W. S. Snyder: *A note on the interiority of real functions.* Preliminary report.

Let F be a continuous real-valued function defined on a space X. Let A be the set of points of X where F has a local maximum and let B denote the set of points of X where F has a local minimum. If X is locally connected then the function (transformation) F is interior at $p \in X$ if and only if $p \in X-(A+B)$. If X has a countable basis then $F(A+B)$ is countable. These results together constitute an extension of a theorem of Whyburn (Bull. Amer. Math. Soc. vol. 48 (1942) pp. 942–945). (Received March 25, 1943.)

148. Otto Szász: *On some trigonometric summability methods and Gibbs phenomenon.*

In this paper certain summability methods of trigonometric type are discussed in relation to Cesàro summability. Applied to Fourier series they lead to new formulae

for the determination of the jump at a point, and to a generalized Gibbs phenomenon. In particular, the Fourier series of an integrable function $f(t)$ presents a Gibbs phenomenon at $\theta=0$, whenever for some $j\neq 0$, $(1/\theta)\int_0^\theta |f(t)-j/2|\,dt\to 0$ as $\theta\to 0$. (Received March 1, 1943.)

149. A. E. Taylor: *Banach spaces of functions representable by Cauchy's integral formula.*

Let $I[\phi; z]$ denote Cauchy's integral $(2\pi i)^{-1}\int \phi(t)(t-z)^{-1}dt$, the integration extended over $|t|=1$ in the counterclockwise sense. Let E^p be the class of functions $f(z)$ representable when $|z|<1$ by $f(z)=I[\phi; z]$, where $\phi(t)$ is an arbitrary function of class L^p on $|t|=1$. Let G^p denote the class of functions representable in the same way when $|z|>1$. Let H^p denote the class of functions $f(z)$ analytic when $|z|<1$, and such that the mean value of $|f(re^{i\theta})|^p$ on $|z|=r$ is bounded when $r<1$. It is assumed that $p>1$. The author proves that E^p and H^p are identical, and that $f(z)$ is in G^p if and only if it is analytic when $|z|>1$, zero at infinity, and such that the mean value of $|f(re^{i\theta})|^p$ on $|z|=r$ is bounded when $r>1$. Any function $F(t)$ of class L^p on $|t|=1$ is representable uniquely in the form $f(t)+g(t)$, almost everywhere on $|t|=1$, where $f\in E^p$ and $g\in G^p$. $I[F; z]$ is equal to $f(z)$ if $|z|<1$, and to $-g(z)$ if $|z|>1$. E^p and G^p are, in a natural sense, complex Banach spaces equivalent to complementary closed linear manifolds M and N, respectively, in L^p. The form of linear functionals on E^p is determined, and it is proved that $(E^p)^*$ is isomorphic with E^q, where $p^{-1}+q^{-1}=1$. A sequence $\{f_n(z)\}$ in E^p converges weakly to $f(z)$ if and only if $f_n(z)\to f(z)$, $|z|<1$, and the mean value of $|f_n(re^{i\theta})|^p$ on $|z|=r$ is bounded uniformly as to n and r, $r<1$. (Received February 23, 1943.)

150. W. J. Thron: *Twin convergence regions for continued fractions.*

Two regions B_0 and B_1 in the complex number plane are called twin convergence regions for the continued fraction $1+K(b_n^2/1)$ if the conditions $b_{2n}\in B_0$ and $b_{2n-1}\in B_1$ (for all $n\geq 1$) insure its convergence. It is shown that the regions $z\in B_0$, if $|z|\leq M$; $z\in B_1$, if $|z-i|\geq M+\epsilon$ and $|z+i|\geq M+\epsilon$, are twin convergence regions for every arbitrary small $\epsilon>0$ and every positive $M<1$. Another family of twin convergence regions is formed by sets B_0 and B_1, if both sets are closed and bounded and if $z=x+iy\in B_0$, if $ax-b<y<ax+b$; and $z\in B_1$, if $ax-1+b<y<ax+1-b$. The number a here can be an arbitrary real number and b any number $0<b<1$. Concerning the bestness of these results the following theorem has been proved. Two regions B_0 and B_1 can be twin convergence regions only if for every $z\in B_0$ the four points $z+i$, $z-i$, $-z+i$, $-z-i$ are not interior points of B_1. These results contain and improve certain earlier results by Leighton and Wall (Amer. J. Math. vol. 38 (1936)), Paydon and Wall (Duke Math. J. vol. 9 (1942)) and Leighton and Thron (Duke Math. J. vol. 9 (1942)). (Received March 1, 1943.)

151. S. M. Ulam: *Theory of the operation of product of sets.* II. Preliminary report.

A projective algebra is a Boolean algebra of sets contained in a direct product E^n of a set E with itself and closed with respect to the operations of projection of a set into the component sets E (considered as a subset of E^n) and the direct product of n sets of the algebra contained in E. A systematic investigation of structures of projective algebras and the properties of projective isomorphisms is undertaken. The

cases of infinite projective algebras generated from a finite number of elements by the Boolean operations and the operation of direct product and projection, and the question of existence of finite bases in the above sense for countable algebras are studied. Particular Boolean algebras studied in connection with these ideas include: the algebra of all sets of integers modulo finite sets, the algebra of sets of integers modulo the sets of density 0, the algebra of finite sums of dyadic intervals on (0, 1), the algebra of Borel sets modulo sets of measure 0. (Received March 25, 1943.)

152. F. A. Valentine: *A Lipschitz condition preserving extension for a vector function.*

The function $f(x)$ mapping a set S in the n-dimensional Euclidean space E_n into a set S' in E_n, and satisfying the condition $|f(x_1)-f(x_2)| \leqq K|x_1-x_2|$ on S, can be extended to any set $T \supset S$ so as to preserve the Lipschitz condition. This is a generalization of a previous result given by the author for the plane (see Bull. Amer. Math. Soc. vol. 49 (1943) p. 100). This extension is a consequence of the following fact: Consider in E_n a set of hyperspheres which have a point in common. Move these hyperpsheres to new positions in E_n, subject to the condition that the distance between any pair of centers is not increased. Then all the hyperspheres in the new positions will still have a point in common. This is first proved for $n+1$ hyperspheres. A theorem of Helly is used to proceed from a set containing $n+1$ hyperspheres to an arbitrary set. Corresponding theorems are obtained for the surface of an n-dimensional hyperhemisphere and for a Hilbert space. (Received March 17, 1943.)

153. H. S. Wall and Marion D. Wetzel: *Contributions to the analytic theory of J-fractions.* Preliminary report.

The authors develop the theory of the class of J-fractions $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2(b_3+z)-\cdots$, $a_p \neq 0$, all of whose approximants $f_p(z)$ satisfy the condition $\Im[f_p(z)]<0$ for $\Im(z)>0$. These J-fractions are completely characterized by the condition: $\sum_{r=1}^{p}\Im(b_r)x_r^2-\sum_{r=1}^{p-1}\Im(a_r)x_rx_{r+1} \geqq 0$, $p=1, 2, 3, \cdots$, for all real x_r. Hence they include the classical J-fractions ($\Im(b_r)=\Im(a_r)=0$) and also those discussed in a recent paper of Hellinger and Wall (Ann. of Math. vol. 44 (1943) pp. 103–127) where $\Im(b_r) \geqq 0$, $\Im(a_r)=0$. It is shown that all the poles of $f_p(z)$ lie in the lower half-plane $\Im(z) \leqq 0$ and there exists a nondecreasing function $\phi_p(u)$, $0 \leqq \phi_p(u) \leqq 1$, such that $f_p(z)=\int_{-\infty}^{\infty} d\phi_p(u)/(z+u)$; therefore the sequence $\{f_p(z)\}$ is uniformly bounded for $\Im(z) \geqq \delta > 0$ and contains an infinite subsequence converging uniformly on the interior of the upper half-plane to an integral of the same form. The older theories, including asymptotic properties and the moment problem, are extended to this general class of J-fractions. (Received March 26, 1943.)

154. Alexander Weinstein: *Differential equations with general boundary conditions.*

The following problem, discussed in this paper, is a typical example of a new kind of differential eigenvalue problems. Let $g_1, g_2, \cdots, g_m$ be a basis of a linear manifold L_m of functions defined on the boundary C of a domain S. It is required to find the eigenvalues of the partial differential equation $\Delta u+\lambda u=0$ with the boundary conditions: u is orthogonal to all elements of L_m and du/dn is an element of L_m. Other examples of the same kind have been introduced by the author in previous papers. The corresponding problems for ordinary differential equations are connected with the classical theory. (Received March 23, 1943.)

APPLIED MATHEMATICS

155. H. R. Branson: *On the difference equation of a general quantum mechanical problem.*

The problems of quantum mechanics may be derived from the classical functions either in the q-language or the p-language. In the q-language $F(q, p) = F(q, hi^{-1}\partial/\partial q)$; in the p-language $F(q, p) = F(-hi^{-1}\partial/\partial p, p)$. With certain types of potential function, the p-language leads to difference equations. Because of the great popularity of quantum mechanics the expression of a general problem in difference equations may aid in "the creation of a usable mathematics of the discrete". The usual equation in one dimension is $H(q, p)\psi = E\psi$ where $H(q, p) = (1/2m)p^2 + V(q)$. The equation becomes in the p-language $(1/2m)p^2\psi + V(-hi^{-1}\partial/\partial p)\psi = E\psi$. Limiting potentials $V(q)$ to those which may be expanded in Fourier series, it follows that: $V(-hi^{-1}\partial/\partial p)\psi = \sum_{k=0}^{\infty}[a_k\psi(p+kh) + b_k\psi(p-kh)]$. With some obvious simplification, the general difference equation for this type of one dimensional quantum mechanical problem may be written in the form $\sum_{k=-\infty}^{\infty} C_k\psi(p+kh) = 0$, wherein all C_i are constants except $C_0 = (p^2/2m + a_0 + b_0 - E)$. Some examples and extensions are discussed. (Received March 25, 1943.)

156. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.* II.

The method outlined in Part I (Bull. Amer. Math. Soc. abstract 49-1-60) is applied for values of $q = 0$ and $q = 1/2$ to the flow of a perfect gas around a circular obstacle. The nature of the variations in the boundary of the flow are discussed, and the effect of compressibility on the pressure is considered. (Received February 22, 1943.)

157. J. F. Harding and Isaac Opatowski: *An approximate formula for the Legendre elliptic integral of the second kind.*

By means of ultraspherical polynomials $E = \int_0^{\pi/2}(1 - k^2\sin^2\phi)^{1/2}d\phi$ is expanded in $E = (1+k')\sum a_n k_1^n$ where $k' = (1-k^2)^{1/2}$, $k_1 = (1-k')/(1+k')$. By a suitable change of the series $\sum a_n k_1^n$ the formula $E = (\pi/4)(1+k')[0.75 + 0.1875k_1^2 + (4-k_1^2)^{-1}] + \epsilon$ is obtained, where $0 \leqq \epsilon \leqq 1 - (61\pi/192) = 0.00189\cdots$, which is a simpler and a closer approximation than the formulas of G. A. Grünberg (Applied Mathematics and Mechanics vol. 1 (1933) pp. 61–69) or those given by J. Thomae (*Formeln und Sätze aus dem Gebiete der elliptischen Funktionen*, Leipzig, 1905, p. 29) or by W. Laska (Sammlung von Formeln, pp. 501–502). (Received March 26, 1943.)

158. Wilfred Kaplan and Max Dresden: *Topology of the molecular N-body problem.*

The N molecules of a gas are considered as mass-particles exerting forces on each other derived from a potential of the form $ar^{-n} - br^{-m}$, $n > m > 0$, $a > 0$, $b > 0$. The force is thus highly repulsive for small r and weakly attractive for large r. The energy integral: Total Kinetic Energy plus Total Potential Energy = Const. $\equiv C$ is then interpreted as restricting the trajectories of the system, in the corresponding $6N$-dimensional phase space, to a $(6N-1)$-dimensional hypersurface $M(C)$. It is shown that for $C \geqq 0$, $M(C)$ has the topological structure of a $(6N-1)$-sphere, minus certain $(6N-4)$-spheres corresponding to collisions. For $-\delta < C < 0$, where δ is a certain explicitly known function of a, b, n, m (not of N), $M(C)$ is homeomorphic to the topo-

logical product of a $(3N-1)$-sphere and a $3N$-sphere, minus certain $(6N-4)$-dimensional loci corresponding to collisions. For $C \leqq -\delta$ the structure of $M(C)$ has yet to be determined. It is planned to correlate the critical values $C=0$ and $C=-\delta$ with the physically known transition temperatures such as the critical temperature. (Received March 23, 1943.)

159. Isaac Opatowski: *An explicit formula for the refractive index in electron optics.*

The refractive index μ is expressed in electron optics (W. Glaser, Zeitschrift für Physik vol. 81 (1933) pp. 647–686) in terms of the electrostatic potential V, the magnetic vector potential $\boldsymbol{A}$ and the unit vector $\boldsymbol{s}$, which is defined as tangent to the electron trajectory. Since $\boldsymbol{s}$ is not known a priori and is a function of V and $\boldsymbol{A}$, the elimination of $\boldsymbol{s}$ from the expression of μ is of advantage. This is done in the paper for a very ample class of fields in which a momentum integral of the equations of motion exists (Bull. Amer. Math. Soc. vol. 46 (1940) p. 887 and Journal of Mathematics and Physics vol. 20 (1941) pp. 418–424). (Received March 26, 1943.)

Geometry

160. Jesse Douglas: *Point transformations and isothermal families of curves.* II.

This paper is a continuation of one with the same title (see Bull. Amer. Math. Soc. abstract 49-1-71). Its new feature is the principal use of synthetic rather than analytic methods. The problem is referred to the investigation of certain properties of a hexagonal web. (Received February 27, 1943.)

161. Jacques Dutka: *Transversality in higher space.*

In this paper, a geometric criterion for transversality developed by Kasner in his paper *Transversality in space of three dimensions* (Trans. Amer. Math. Soc. vol. 30 (1928) pp. 447–452) is generalized for n-dimensional Euclidean space. It is shown here that a necessary and sufficient condition for a given correspondence between a lineal element and a hypersurface element to be a transversality is that a certain induced correlation be a polarity. A principle of transference connecting simple and $(n-1)$-fold integrals in the calculus of variations when they produce equivalent transversalities is established. The result obtained is applied to the theory of infinitesimal contact transformations from which are derived analytic tests equivalent to the above-mentioned geometric criterion. Actual examples of transversalities in addition to the well known condition of orthogonality are also given. (Received March 25, 1943.)

162. Jacques Hadamard: *On fractional iteration and connected questions.*

The author presents some results communicated to him by two younger geometers on fractional iteration and permutable transformations in one variable. This subject is connected with group theory or, more precisely, with Cartan's conception of geodesics in a group-space. (Received March 27, 1943.)

163. T. R. Hollcroft: *Plane curve systems with distinct nodes and cusps and of negative virtual dimension.*

In this paper a system C of plane curves is defined by plane sections of a tangent cone of species $r-2$ from an S_{r-3} to a nonsingular, irreducible primal V_{r-1} of order ν in S_r. The only singularities of C are distinct nodes and cusps. The system C has a negative virtual dimension for certain limiting values of r and ν. The above has been presented to the Society (Bull. Amer. Math. Soc. abstract 44-11-412). At that time, the system C had not been proved irreducible. The irreducibility of C has been established. These systems are at present the only plane curve systems known to exist of negative virtual dimension with no other singularities than distinct nodes and cusps. (Received March 26, 1943.)

164. Edward Kasner and John DeCicco: *Generalized transformation theory of isothermal families.*

Kasner has proved that the complete group of lineal-element transformations which send every isothermal family of curves into an isothermal family is the product of the conformal group by the non-contact group $U=u$, $V=v$, $\Theta=a\theta+h(u)+k(v)$. (Note that (u, v) are the minimal coordinates of the point and θ is the inclination.) Further generalizations have been given by the authors to first order field-element transformations and curvature-element transformations (Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 406–412, vol. 28 (1942) pp. 52–55 and pp. 328–338). In this paper, the authors determine all nth order field-element to lineal-element transformations which preserve the isothermal character. Denote $r_\alpha=\partial^\alpha\theta/\partial u^\alpha$, $l_\beta=\partial^\beta\theta/\partial v^\beta$, $s_{\alpha\beta}=\partial^{\alpha+\beta}\theta/\partial u^\alpha\partial v^\beta$ for $\alpha, \beta=1, 2, \cdots, n$. Any transformation of our set must be of the form $U=\phi(u, r_\alpha, s_{\alpha\beta})$, $V=\psi(v, s_{\alpha\beta}, l_\beta)$, $\Theta=a\theta(s_{\alpha\beta})+h(u, r_\alpha, s_{\alpha\beta})+k(v, s_{\alpha\beta}, l_\beta)$, or this type followed by a reflection in the X-axis. Finally it is shown that the total set of all nth order differential-element to lineal-element transformations preserving the class of all isothermal families is exactly the Kasner group. (Received March 23, 1943.)

165. Edward Kasner and John DeCicco: *The congruence of element-series associated with a polygenic function.*

The derivative $\gamma=dw/dz$ of the polygenic function w with respect to z induces a correspondence T between the lineal-elements (x, y, θ) of the z-plane and the points $\gamma=\alpha+i\beta$ of the γ-plane. This associated transformation T carries the ∞^1 elements at a point in the z-plane into a circle (the Kasner circle) in the γ-plane. However, a point in the γ-plane does not correspond to a single element but to a series of elements. Therefore, in general, there exists a congruence (∞^2) of series in the z-plane which by T is converted into the set of ∞^2 points in the γ-plane. Kasner has already determined those polygenic functions for which the associated ∞^2 series are all unions. In that event, the unions are all circles passing through a fixed point (Bull. Amer. Math. Soc. vol. 44 (1938) pp. 726–732). In this paper, a complete analytic characterization of the congruence of element-series associated with any polygenic function is given. One geometric property is that, if the tangent turbines of the ∞^1 series which pass through a given point z are constructed at z, the centers of these turbines will describe a conic section. (March 23, 1943.)

166. J. E. Wilkins: *A special class of surfaces in projective differential geometry.*

It is known that the asymptotic osculating quadrics at a point of a curve on a surface coincide if and only if the curve is tangent to a curve of Darboux and the surface satisfies the relation (1) $\beta\psi^3=\gamma\phi^3$. This paper investigates the properties of surfaces which satisfy this relation indentically. In particular coincidence surfaces possess all of these properties. There exist, however, surfaces satisfying the relation (1) which are not coincidence surfaces. A necessary and sufficient condition that a surface which satisfies (1) be also a cubic surface is given and used to prove that there is, in the sense of projective equivalence, only one cubic coincidence surface. (Received March 11, 1943.)

Logic and Foundations

167. B. A. Bernstein: *Postulate sets for Boolean rings.*

The author gives nine sets of postulates for Boolean rings in terms of ring operations. Each set is independent, and remains independent when a unit-element postulate is added. (Received March 25, 1943.)

168. Max Zorn: *Informal note on the second underivability theorem.*

The content of this note may be condensed into one question: In which sense does a formula like (Ex) (Form (x) & $\overline{\text{Bew}}$ (x)), which from the formalist point of view has no independent meaning, "represent" the consistency of a formalism in the sense of Hilbert? The answer to this question is expected from those who insist that the underivability of such formulae constitutes evidence in support of the opinion that finitary consistency proofs of the type which so far have been employed by the Hilbert school probably cannot be found for the arithmetic formalism or *Principia Mathematica*. (Received March 27, 1943.)

Statistics and Probability

169. Leon Alaoglu: *Harmonic analysis of stochastic processes.* Preliminary report.

If Ω is a differential stochastic process of elements $z(t)$ which are complex-valued functions of a real variable t, such that the distribution function of the variable $e^{i\theta}(z(t+h)-z(t))$ is independent of θ (θ real) and if the expectations $F(t)=\int_\Omega |z(t)-z(0)|^2 dP$, $m(t)=\int_\Omega (z(t)-z(0))dP$ exist, the first being bounded and the second vanishing, then the Fourier transform of the function $z(t)$ exists for almost all z and defines a stationary stochastic process. (Received March 26, 1943.)

Topology

170. B. H. Arnold: *On decompositions of T_1 spaces.*

Several authors (Banach, *Théorie des Opérations Linéaires*, p. 170; Eilenberg, Ann. of Math. vol. 43 (1942) pp. 568–579; Eidelheit, Studia Mathematica vol. 9 (1940) pp. 97–105) have proved theorems of the form: The "structure" of a certain class of transformations defined on a suitable space A to a fixed suitable space B determines the space A. In the present paper the author proves an analogous result which is valid for a very wide class of spaces A, but at the expense of allowing B to become variable. If two T_1 spaces, A, A' are such that the ordered system M of the upper semicontinuous decompositions of A is isomorphic to that of A', then A and A' are homeomorphic. Separation, connectedness, and compactness properties of the space

A are expressed in terms of order properties of M and a procedure is given which allows A to be constructed from M, again using only order properties of M. (Received March 6, 1943.)

171. D. G. Bourgin: *On a quasi linear operation.*

Let E be a linear topological space which is convex (though this last restriction can be weakened). The operators referred to below are on a subset of E to E, satisfy $U(\alpha_1 x+\alpha_2 y)=\alpha_1 U(x)+\alpha_2 U(y)$ for $\alpha_i \geqq 0$, $\alpha_1+\alpha_2=1$, and are continuous. A typical result is that if ϕ induces an automorphism of the convex bicompact set $K \subset E$ then the relation $\phi=(\psi+\gamma)/2$ where $\psi(K)=K$ implies $\gamma=\psi=\phi$. An incidental result is that under certain fairly general conditions U may be extended to U' on the closed linear extension of the original domain and then satisfies $U'(ax+by)=aU'(x)+bU'(y)+(a+b-1)U'(\theta)$ and is merely a translation plus a linear transformation. (Received March 26, 1943.)

172. R. H. Fox: *Natural systems of homomorphisms.* Preliminary report.

Let P_{2k+1} denote the kth homotopy group of a complex Y, P_{2k+2} the kth homotopy group of the n-dimensional skeleton X, and P_{2k} the kth homotopy group of Y mod X. Let Q_{2k+1}, Q_{2k+2} and Q_{2k} denote the corresponding continuous homology groups with integer coefficients. The groups P_m, Q_m, together with the natural homomorphisms $r=r_m^{(P)}$ of P_m into P_{m-1}, $r=r_m^{(Q)}$ of Q_m into Q_{m-1}, and $h=h_m$ of P_m into Q_m constitute the natural system of Y, X. The nucleus of r_m is the image of r_{m+1}, and $hr=rh$. These two simple facts lead to the following theorem: If Y is simply connected and acyclic in the dimensions less than or equal to $n-2$, then every n-cycle is spherical. Other consequences are a generalization of a theorem of S. Eilenberg (Bull. Amer. Math. Soc. vol. 47 (1941) p. 432) and the recent theorem of H. Hopf (Comment. Math. Helv. vol. 14 (1942) p. 257). (Received March 25, 1943.)

173. R. C. James: *Orthogonality in normed linear topological spaces.*

The purpose of this paper is to investigate certain possible definitions of orthogonality in normed linear topological spaces, properties possessed by elements orthogonal in each sense, and conditions that each type of orthogonality have certain common and desirable properties. These properties are: (1) If x is orthogonal to $y (x \perp y)$, then $y \perp x$; (2) if $x \perp y$, then $ax \perp by$ for all a and b; (3) if $x \perp y$ and $x \perp z$, then $x \perp (ay+bz)$ for all a and b; (4) for any elements x and y there exists a number a such that $x \perp (ax+y)$. The three definitions of orthogonality which are used are: (1) $x \perp y$ if $\|x-y\|=\|x+y\|$; (2) $x \perp y$ if $\|x\|^2+\|y\|^2=\|x-y\|^2$; (3) $x \perp y$ if $\|x+ay\| \geqq \|x\|$ for all a. All three types of orthogonality have the fourth property, while it is shown that if orthogonality of either of the first two definitions has all four properties the norm can be defined by a bilinear and symmetric inner product. All three definitions are equivalent and have all four properties when the norm is given by such an inner product. (Received March 12, 1943.)

174. Tibor Radó: *On continuous path-surfaces of zero area.*

Let $S: x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v) \in Q: 0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, be a continuous path-surface. Let $[S]$ denote the point-set in (x, y, z) space that corresponds to Q by means of the equations of S. The Lebesgue area $A(S)$ of S is not determined by

$[S]$, but rather by the manner in which $[S]$ is described while (u, v) describes Q. In fact, simple examples, due to Geöcze, show that $A(S)$ may equal zero even though $[S]$ is a solid cube. Geöcze derived various necessary and sufficient conditions for $A(S)=0$. The purpose of this paper is to establish further conditions of this type. The methods used are essentially topological. (Received February 19, 1943.)

175. G. E. Schweigert: *Fixed elements and periodic types for homeomorphisms on semi-locally-connected continua.*

Let S be a semi-locally-connected continuum and $T(S)=S$ a homeomorphism. If $N\neq S$ is an invariant node then there exists in S another invariant cyclic element $E\neq N$. This leads to equivalence of five properties of Ayres and Whyburn (cf. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications vol. 28, 1942, chap. 12, Theorems 4.21–4.5); pointwise almost periodicity is disregarded, second property "localized" to invariant A-set. Componentwise periodicity means, for every true invariant A-set A, each component of $S-A$ has finite period. In order that $T(S)=S$ be elementwise periodic on cyclic elements of complement of end points of S it is necessary and sufficient that one of the following conditions holds: (a) T is componentwise periodic, and for every positive integer n, T^n has one of five properties above; (b) T is componentwise periodic and each T^n-invariant A-set A, with no points of A on true cyclic element of $S-A$, admits a contracting T^n-approximation at A, that is, for every A-set C with $A\subset \text{Int } C$ there is a third T^n-invariant A-set B such that $A\subset \text{Int } B\subset C$; (c) Theorem (4.7), chap. 12, holds. Part (b) is applied to give a certain continuous orbit decomposition. (Received February 27, 1943.)

176. G. E. Schweigert: *Minimal A-sets, infinite orbits, and fixed elements.*

If E is a cyclic element of a semi-locally-connected continuum S with infinite period relative to the homeomorphism $T(S)=S$ and B is the least invariant A-set which contains the orbit of E, then $B=C(A, D)+0$, where 0 is the closure of the orbit of a cyclic chain $C(E, Y)$ with $C(E, Y)\cdot C(A, D)=Y$ and A and D are the invariant cyclic elements of B determining the cyclic chain $C(A, D)$. If $A\neq D$, then A and D are fixed end points of $C(A, D)$; if $A=D$, then $C(A, D)=Y$ is an unique invariant cyclic element called center of rotation. If $E=Y$, then $B=C(A, D)$. As a corollary it is sufficient to get the Ayres' property (Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications vol. 28, 1942, chap. 12, Theorem 4.5) to assume $C(E, T(E))$ contains an invariant element in (and only in) case E has an infinite orbit. If $T(S)=S$ is elementwise periodic on all cyclic elements except the end points of S, the least invariant A-set M containing the infinite orbit of an end point p of S has an unique invariant element, and the dendritic form of the cyclic element hyperspace of M is similar to the dyadic tree. (Received February 27, 1943.)

177. J. W. T. Youngs: *On parametric representations of surfaces.* III.

A representation is a continuous transformation $R(A)=B$ from the 2-sphere A to a set B in 3-space. $R(A)=L(M(A))$, where $M(A)=\Sigma$ is monotone and $L(\Sigma)=B$ is light. Σ is imbedded in 3-space so that R is of degree 1 on each true cyclic element of Σ. Two representations R_1 and R_2 are F^+ equivalent if the Fréchet distance between them is zero and the homeomorphisms of the definition are of degree $+1$. They are K^+ equivalent if there is a homeomorphism $H(\Sigma_1)=\Sigma_2$ such that $L_1(\Sigma_1)=L_2(H(\Sigma_1))$

and H is of degree $+1$ on each true cyclic element of Σ_2. F^- and K^- equivalence are defined similarly. R_1 is Fréchet equivalent to R_2 if R_1 is either F^+ or F^- equivalent to R_2, and similarly for K equivalence. Various relations between these definitions are investigated. Applications to the theory of area are made on the basis of the fact that F equivalence is the same as K equivalence. A general method for obtaining smooth representations is developed. The content of the paper is examined from the point of view of the μ-length of Morse. (Received March 26, 1943.)

NEW PUBLICATIONS

CARNAP, R. Formalization of logic. Cambridge, Harvard University Press; London, Humphrey Milford, 1943. 18+159 pp. $3.00.

DAVIS, D. S. Empirical equations and nomography. New York and London, McGraw-Hill, 1943. 9+200 pp. $2.50.

JUSTICE, H. K. See Smith, E. S.

LEVI, F. W. Algebra. Vol. 1. Calcutta, University of Calcutta, 1942. 12+305 pp.

REDDICK, H. W. Differential equations. New York, Wiley; London, Chapman and Hall, 1943. 9+245 pp. $2.50.

SALKOVER, M. See Smith, E. S.

SMITH, E. S., SALKOVER, M., and JUSTICE, H. K. Analytic geometry. New York, Wiley; London, Chapman and Hall, 1943. 12+298 pp. $2.50.

THOMPSON, A. J. Table of the coefficients of Everett's central-difference interpolation formula. 2d edition. (Tracts for Computers, no. 5.) Cambridge University Press 1943. 8+32 pp. 5s.

WINTNER, A. Eratosthenian averages. Baltimore, Waverly, 1943. 6+82 pp. $2.00.